UNIFORM CONVERGENCE THEOREM FOR THE $H_1$-INTEGRAL
REVISITED

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Abstract. In this note we show that the uniform convergence theorem for the $H_1$-integral is false.

1. INTRODUCTION

Most notions and notations we use is taken from [1]. Moreover for partial divisions $D_1$ and $D_2$ of a compact interval $[a,b]$ we write $D_2 \supseteq D_1$, if for every $([s,t], \eta) \in D_2$ there is $([u,v], \xi) \in D_1$ such that $[s,t] \subseteq [u,v]$. Recall from [2] that a function $f: [a,b] \to \mathbb{R}$ is $H_1$-integrable on $[a,b]$ to a number $A \in \mathbb{R}$, if there exists a positive function $\varepsilon$ on $[a,b]$ such that for every $\varepsilon > 0$ there exists a division $D_0$ of $[a,b]$ such that $\left|\sum (D) f(\xi)(v-u) - A\right| < \varepsilon$ for every $\delta$-fine division $D \supseteq D_0$. (This definition is equivalent to the one from [1], see [3, Theorem 2.4]. The difference is in the definition of the relation $\supseteq$.) In this case we will say that $A$ is the $H_1$-integral of $f$ on $[a,b]$ and write $A = \int_a^b f$.

I. J. L. Garces and P. Y. Lee claimed to prove the uniform convergence theorem for the $H_1$-integral [1, Theorem 4]. Alas, we have the following theorem.

Theorem 1. The uniform convergence theorem does not hold for the $H_1$-integral.

Proof. Let $(G_n)$ be a sequence of open dense subsets of $[0,1]$, whose intersection $E$ is a null set. For each $n \in \mathbb{N}$ denote by $h_n$ the characteristic function of $[0,1] \setminus G_n$. By [2, Lemma 4], each function $h_n$ is $H_1$-integrable.
The series \( f = \sum_{n \in \mathbb{N}} h_n/2^n \) is uniformly convergent on \([0, 1] \). Suppose that \( f \) is \( H_1 \)-integrable using a positive function \( \delta \) on \([0, 1] \). For each \( n \in \mathbb{N} \) let
\[
E_n = \{ x \in E : \delta(x) > n^{-1} \}.
\]
The set \( E \) is a dense \( \mathcal{G}_d \) set, so it is residual. Thus there is an \( n \in \mathbb{N} \) and an open interval \( I \) such that \( E_n \) is dense in \( I \). Since \( E \) is a null set and \( f > 0 \) outside of \( E \), there is an \( m \geq n \) such that the measure of the closure of the set
\[
F_m = \{ x \in I \setminus E : f(x) > m^{-1} \text{ and } \delta(x) > m^{-1} \}
\]
is positive, say \( M \).

Define \( \varepsilon = M/(2m) \). By assumption, there is a division \( D_0 \) of \([0, 1] \) such that
\[
|(D) \sum f(\xi)(v - u) - \int_a^b f| < \varepsilon
\]
for every \( \delta \)-fine division \( D \supseteq D_0 \). Without loss of generality we may assume that a subset of \( D_0 \), say \( D_1 \), is a division of \( I \). Every interval from \( D_1 \) can be written as the union of a finite family of nonoverlapping intervals of length less than \( m^{-1} \).

Denote by \( \mathcal{A} \) the family of all these intervals. Clearly if
\[
\mathcal{B} = \{ J \in \mathcal{A} : J \cap F_m \neq \emptyset \},
\]
then \( \sum_{J \in \mathcal{B}} |J| \geq M \).

For each \( J \in \mathcal{B} \) we can pick an \( x_J \in J \cap F_m \) and, as \( E_n \) is dense in \( I \), a \( y_J \in J \cap E_n \). Let \( D_1 \supseteq D_0 \) be a \( \delta \)-fine division of \([0, 1] \setminus \bigcup_{J \in \mathcal{B}} J \). Both \( D_2 = \{ (J, x_J) : J \in \mathcal{B} \} \) and \( D_3 = \{ (J, y_J) : J \in \mathcal{B} \} \) are \( \delta \)-fine partial divisions of \([0, 1] \) and \( D_2, D_3 \supseteq D_1 \). So, \( D_4 = D_1 \cup D_2 \) and \( D_5 = D_1 \cup D_3 \) are \( \delta \)-fine divisions of \([0, 1] \) such that \( D_4, D_5 \supseteq D_0 \).

For all \( J \in \mathcal{B} \) we have \( f(x_J) > m^{-1} \) and \( f(y_J) = 0 \). Thus
\[
(D_4) \sum f(\xi)(v - u) - (D_5) \sum f(\xi)(v - u) > m^{-1} \sum_{J \in \mathcal{B}} |J| \geq M/m = 2\varepsilon,
\]
contrary to (1). Consequently, \( f \) is not \( H_1 \)-integrable.

I. J. L. Garces and P. Y. Lee proved the controlled convergence theorem for the \( H_1 \)-integral \([1, \text{Theorem } 6] \): if an equi-\( H_1 \)-integrable sequence of functions \( (f_n) \) is pointwise convergent to some function \( f \), then \( f \) is \( H_1 \)-integrable and \( \int f = \lim f_n \).

In view of Theorem 1, we obtain the following corollary.

**Corollary 2.** There is a uniformly convergent sequence of \( H_1 \)-integrable functions which is not equi-\( H_1 \)-integrable.

So, \([1, \text{Theorem } 4] \) (the one which turned out to be false) does not follow from \([1, \text{Theorem } 6] \), contrary to the remark at the bottom of page 444 of \([1] \).
REFERENCES


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