INTEGRAL REPRESENTATIONS FOR SRIVASTAVA’S TRIPLE HYPERGEOMETRIC FUNCTIONS

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Abstract. While investigating the Lauricella’s list of 14 complete second-order hypergeometric series in three variables, Srivastava noticed the existence of three additional complete triple hypergeometric series of the second order, which were denoted by $H_A$, $H_B$ and $H_C$. Each of these three triple hypergeometric functions $H_A$, $H_B$ and $H_C$ has been investigated extensively in many different ways including, for example, in the problem of finding their integral representations of one kind or the other. Here, in this paper, we aim at presenting further integral representations for each of Srivastava’s triple hypergeometric functions $H_A$, $H_B$ and $H_C$.

1. INTRODUCTION AND PRELIMINARIES

In the theory of hypergeometric functions of several variables, a remarkably large number of triple hypergeometric functions have been introduced and investigated. A comprehensive table of 205 distinct triple hypergeometric functions is provided in the work of Srivastava and Karlsson [13, Chapter 3]. Out of these 205 distinct triple hypergeometric functions, Lauricella [7, p. 114] introduced fourteen complete triple hypergeometric functions of the second order. He denoted his triple hypergeometric functions by the symbols $F_1, \cdots , F_{14}$ of which $F_1, F_2, F_5$ and $F_9$ correspond, respectively, to the three-variable Lauricella functions $F_A^{(3)}, F_B^{(3)}, F_C^{(3)}$ and $F_D^{(3)}$ that are the three-variable cases of the $n$-variable Lauricella functions $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}$ and $F_D^{(n)}$ (cf. [7, p. 113]; see also [1, p. 114, Equations (1) to (4)], [13, p. 33 et seq.] and [4, 5]). Saran [9] initiated a systematic study of ten of the triple
hypergeometric functions from Lauricella’s set. Exton [3] introduced 20 distinct triple hypergeometric functions, which he denoted by \( X_1 \cdots X_{20} \), and investigated their twenty Laplace integral representations whose kernels include the confluent hypergeometric functions \( {}_0F_1 \) and \( {}_1F_1 \), and the Humbert hypergeometric functions \( \Phi_2 \) and \( \Psi_2 \) of two variables. The four Appell hypergeometric functions \( F_1, \cdots F_4 \) of two variables are simply the special case of Lauricella’s \( n \)-variable functions when \( n = 2 \), that is,

\[
F_1 = F_D^{(2)}, \quad F_2 = F_A^{(2)}, \quad F_3 = F_B^{(2)} \quad \text{and} \quad F_4 = F_C^{(2)}.
\]

While transforming Pochhammer’s double-loop contour integrals associated with the functions \( F_5 \) and \( F_{14} \) (that is, \( F_C \) and \( F_F \), respectively) belonging to Lauricella’s set of hypergeometric functions of three variables, Srivastava [10, 11] discovered the existence of three additional complete triple hypergeometric functions \( H_A, H_B \) and \( H_C \) of the second order, which are defined as follows (see also [13, p. 43, Equations 1.5(11) to 1.5(13)]):

\[
H_{A}(a_1, a_2, a_3; c_1, c_2; x, y, z)
= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+p}(a_2)_{m+n}(a_3)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m y^n z^p}{m! n! p!}
\]

\[
(0 < \tau < 1; \quad |y| < 1; \quad |z| < (1 - \tau)(1 - \sigma))
\]

\[
H_{B}(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z)
= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+p}(a_2)_{m+n}(a_3)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m y^n z^p}{m! n! p!}
\]

\[
(\nu := |x|; \quad \sigma := |y|; \quad \tau := |z|; \quad \nu + \sigma + \tau + 2\sqrt{\nu\tau\sigma} < 1)
\]

and

\[
H_{C}(a_1, a_2, a_3; c; x, y, z)
= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+p}(a_2)_{m+n}(a_3)_{n+p}}{(c)_{m+n+p}} \frac{x^m y^n z^p}{m! n! p!}
\]

\[
(\nu := |x|; \quad \sigma := |y|; \quad \tau := |z|; \quad \nu + \sigma + \tau - 2\sqrt{(\nu-\sigma)(\nu-1)(\tau-1)} < 2),
\]

where, with \( \mathbb{C} \) and \( \mathbb{Z}_0^- \) denoting the set of complex numbers and the set of nonpositive integers, respectively, \((\lambda)_n\) is the Pochhammer symbol defined (for \( \lambda \in \mathbb{C} \)) by

\[
(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 
1 & (n = 0) \\
\lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \cdots\})
\end{cases}
\]
\( \Gamma(z) \) being the well-known Gamma function. Of course, all 20 of Exton’s triple hypergeometric functions \( X_1, \ldots, X_{20} \) as well as Srivastava’s triple hypergeometric functions \( H_A, H_B \) and \( H_C \) are included in the set of the aforementioned 205 distinct triple hypergeometric functions which were presented systematically by Srivastava and Karlsson [13, Chapter 3]. The above-stated three-dimensional regions of convergence of the triple hypergeometric series in (1.1), (1.2) and (1.3) for \( H_A, H_B \) and \( H_C \), respectively, were given by Srivastava [10, 11] (see also Srivastava and Karlsson [13, Section 3.4]).

Various multivariable generalizations and cases of reducibility of Srivastava’s functions \( H_A, H_B \) and \( H_C \) have been investigated (see, for details, [13, pp. 43–44]). Turaev [15] studied the Srivastava function \( H_A \). Hasanov et al. [6] reproduced Srivastava’s integral representations [10, 11] for the functions \( H_A, H_B \) and \( H_C \) in the following (potentially useful) forms:

\[
H_A(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2; x, y, z) = \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\gamma_1)\Gamma(\gamma_2)\Gamma(\gamma_1 - \beta_1)\Gamma(\gamma_2 - \beta_2)} \int_0^1 \int_0^1 \xi^{\beta_1-1} \eta^{\beta_2-1} (1-\xi)^{\gamma_1-\beta_1-1}(1-\eta)^{\gamma_2-\beta_2-1}(1-\eta)^{-\beta_1}(1-x\xi-z\eta)^{-\alpha} \cdot \left(1 - \frac{xy\xi\eta}{(1-\eta)(1-x\xi-z\eta)}\right)^{-\alpha} d\xi d\eta
\]

\[
(\Re(\gamma_1) > \Re(\beta_1) > 0; \Re(\gamma_2) > \Re(\beta_2) > 0);
\]

\[
H_B(\alpha, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; x, y, z)
\]

\[
= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^\infty \int_0^\infty e^{-s-t-u} s^{\alpha-1} u^{\beta_1-1} v^{\beta_2-1} F_0(\gamma_1; xst) F_0(\gamma_2; yus) F_0(\gamma_3; zvt) ds dt du
\]

\[
(\min \{\Re(\alpha), \Re(\beta_1), \Re(\beta_2)\} > 0),
\]

where each of the confluent hypergeometric functions \( F_1 \) can be rewritten in terms of the Bessel function \( J_\nu(z) \) and \( I_\nu(z) \) given by

\[
J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu + 1)} F_0(\nu; \nu + 1; -\frac{z^2}{4})
\]

and

\[
I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu + 1)} F_0(\nu; \nu + 1; \frac{z^2}{4}),
\]

respectively;
Here, in this present sequel to some of the above-mentioned works, we aim at investigating some further integral representations for each of the three Srivatava functions $H_A$, $H_B$, and $H_C$.

2. INTEGRAL REPRESENTATIONS OF $H_A$

**Theorem 1.** Each of the following integral representations for $H_A$ holds true:

$$H_A (a_1, a_2, a_3; c_1, c_2; x, y, z) = \frac{\Gamma(c_2)}{\Gamma(a_3) \Gamma(c_2 - a_3)} \int_0^1 \xi^{a_3-1} (1 - \xi)^{c_2-a_3-1}$$

$$\cdot (1 - y\xi)^{-a_2} (1 - z\xi)^{-a_1} \ _2F_1 \left( a_1, a_2; c_1; \frac{x}{(1 - y\xi)(1 - z\xi)} \right) d\xi \quad (\Re(c_2) > \Re(a_3) > 0);$$

$$H_A (a_1, a_2, a_3; c_1, c_2; x, y, z)$$

$$= \frac{\Gamma(c_2) (1 + \lambda)^{a_3}}{\Gamma(a_3) \Gamma(c_2 - a_3)} \int_0^1 \xi^{a_3-1} (1 - \xi)^{c_2-a_3-1}$$

$$\cdot (1 + \lambda\xi)^{a_1+a_2-c_2} [1 + \lambda\xi - (1 + \lambda) \xi y]^{a_2} [1 + \lambda\xi - (1 + \lambda) \xi z]^{-a_1}$$

$$\cdot \ _2F_1 \left( a_1, a_2; c_1; \frac{x (1 + \lambda\xi)^2}{[1 + \lambda\xi - (1 + \lambda) \xi y] [1 + \lambda\xi - (1 + \lambda) \xi z]} \right) d\xi \quad (\Re(c_2) > \Re(a_3) > 0; \lambda > -1);$$

$$H_A (a_1, a_2, a_3; c_1, c_2; x, y, z)$$

$$= \frac{\Gamma(c_2) (\beta - \gamma)^{a_3} (\alpha - \gamma)^{c_2-a_3}}{\Gamma(a_3) \Gamma(c_2 - a_3) (\beta - \alpha)_{c_2-a_3-1}}$$

$$\int_0^\beta (\beta - \xi)^{c_2-a_3-1} (\xi - \alpha)^{a_3-1} (\xi - \gamma)^{a_1+a_2-c_2}$$

$$\cdot [(\beta - \alpha) (\xi - \gamma) - (\beta - \gamma) (\xi - \alpha)] y^{a_2} [(\beta - \alpha) (\xi - \gamma) - (\beta - \gamma) (\xi - \alpha)] z^{-a_1}$$

$$\cdot \ _2F_1 \left( a_1, a_2; c_1; \sigma x \right) d\xi \quad (\Re(c_2) > \Re(a_3) > 0; \gamma < \alpha < \beta),$$

where $x, y, z, \alpha, \beta, \gamma$ are chosen as positive real numbers.
where
\[
\sigma := \frac{(\beta - \alpha)^2 (\xi - \gamma)^2}{[(\beta - \alpha)(\xi - \gamma) - (\beta - \gamma)(\xi - \alpha)] [\beta - \gamma)(\xi - \alpha) z].
\]

\[
H_A (a_1, a_2, a_3; c_1, c_2; x, y, z)
\]
(2.4)
\[
= \frac{\Gamma (e_2)}{\Gamma (a_3) \Gamma (e_2 - a_3)} \int_{0}^{\infty} \xi^{a_3 - 1} (1 + \xi)^{a_1 + a_2 + c_2}
\cdot (1 + \xi - y \xi)^{-a_2} (1 + \xi - z \xi)^{-a_1}
\cdot \frac{\gamma \cdot 2F1 \left( \begin{array}{c} a_1, a_2; c_1; \\ x (1 + \xi) \end{array}; \frac{(1 + \xi - y \xi)(1 + \xi - z \xi)}{(1 + \xi - y \xi + z \xi)} \right) \} d\xi
\]
\[
(\Re (e_2) > \Re (a_3) > 0);
\]

\[
H_A (a_1, a_2, a_3; c_1, c_2; x, y, z)
\]
(2.5)
\[
= \frac{2 \Gamma (e_2)}{\Gamma (a_3) \Gamma (e_2 - a_3)} \int_{0}^{\infty} \xi^{a_3 - 1} (1 + \xi)^{a_1 + a_2 + c_2}
\cdot (1 - y \sin^2 \xi)^{-a_2} (1 - z \sin^2 \xi)^{-a_1}
\cdot \frac{\gamma \cdot 2F1 \left( \begin{array}{c} a_1, a_2; c_1; \\ x \end{array}; \frac{(1 - y \sin^2 \xi)(1 - z \sin^2 \xi)}{(1 - y \sin^2 \xi + z \sin^2 \xi)} \right) \} d\xi
\]
\[
(\Re (e_2) > \Re (a_3) > 0).
\]

Here \(2F1\) denotes the well-known Gauss hypergeometric function defined by
\[
2F1 (a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n
\]
(2.6)
\[
(c \in \mathbb{C} \setminus \mathbb{Z}_0^-; |z| < 1; |z| = 1 (z \neq -1) \quad \Re (c - a - b) > 0;
\]
\[
z = -1 \quad \text{and} \quad \Re (c - a - b) > -1).
\]

Proof. The integral representation (2.1) was derived by Srivastava himself [10, p. 100] as an intermediate result in his demonstration of the integral representation (1.5) [10, p. 100, Equation (3.3)]. In fact, Srivastava’s derivation of (2.1) involved writing the triple hypergeometric series in (1.1) as a single series of the Appell function \(F1\) as follows:

\[
H_A (a_1, a_2, a_3; c_1, c_2; x, y, z)
\]
(2.7)
\[
= \sum_{m=0}^{\infty} \frac{(a_1)_m (a_2)_m}{(c_1)_m m!} F1 [a_3, a_2 + m, a_1 + m; c_2; y, z] \frac{X^m}{m!}
\]
and then applying Picard’s integral formula [1, p. 29, Equation (4)]:

\[
F1 [\alpha, \beta; \gamma; x, y] = \frac{\Gamma (\gamma)}{\Gamma (\alpha) \Gamma (\gamma - \alpha)}
\]
(2.8)
\[
\cdot \int_{0}^{1} \tau^{\alpha - 1} (1 - \tau)^{\gamma - \alpha - 1} (1 - x \tau)^{-\beta} (1 - y \tau)^{-\beta'} d\tau
\]
to each term on the right-hand side of (2.7). The transition from (2.1) to Srivastava’s final result (1.5) was made by appealing to the following classical result (see, for details, [10, pp. 99–100]):

\[
\text{ℜ}(\gamma) > \text{ℜ}(\alpha) > 0; \quad \gamma \in \mathbb{C} \setminus \mathbb{Z}_0.
\]

The assertions (2.4) and (2.5) of Theorem 1 would follow from Srivastava’s result (2.1) upon setting

\[
\xi \mapsto \frac{\xi}{1 + \xi}, \quad d\xi \mapsto \frac{d\xi}{(1 + \xi)^2} \quad \text{and} \quad (0, 1) \mapsto (0, \infty)
\]

and

\[
\xi \mapsto \sin^2 \xi, \quad d\xi \mapsto 2\sin \xi \cos \xi \, d\xi \quad \text{and} \quad (0, 1) \mapsto \left(0, \frac{\pi}{2}\right),
\]

respectively.

Each of the integral representations (2.1) to (2.5) can also be proved directly by expressing the series definition of the involved hypergeometric function \( _2F_1 \) in each integrand and changing the order of the integral sign and the summation, and finally using one or the other of the following well-known relationships between the Beta function \( B(\alpha, \beta) \), the Gamma function \( \Gamma(z) \) and their various associated Eulerian integrals (see, for example, [2, pp. 9–11] and [14, p. 26 and p. 86, Problem 1]):

\[
B(\alpha, \beta) = \begin{cases} 
\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \, dt & (\min\{\text{ℜ}(\alpha), \text{ℜ}(\beta)\} > 0) \\
\frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} & (\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0),
\end{cases}
\]

\[
B(\alpha, \beta) = 2 \int_0^{\frac{\pi}{2}} (\sin \theta)^{2\alpha-1} (\cos \theta)^{2\beta-1} \, d\theta = \int_0^\infty \frac{\tau^{\alpha-1}}{(1 + \tau)^{\alpha + \beta}} \, d\tau
\]

(\min\{\text{ℜ}(\alpha), \text{ℜ}(\beta)\} > 0)

and

\[
B(\alpha, \beta) = \frac{(b-c)^{\alpha}(a-c)^{\beta}}{(b-a)^{\alpha+\beta-1}} \int_a^b \frac{(t-a)^{\alpha-1}(b-t)^{\beta-1}}{(t-c)^{\alpha+\beta}} \, dt \quad (c < a < b)
\]

\[
= (1 + \lambda)^\alpha \int_0^1 \frac{\tau^{\alpha-1}(1-\tau)^{\beta-1}}{(1 + \lambda \tau)^{\alpha+\beta}} \, d\tau \quad (\lambda > -1)
\]

(\min\{\text{ℜ}(\alpha), \text{ℜ}(\beta)\} > 0).
Integral Representations for Srivastava’s Triple Hypergeometric Functions $H_A$, $H_B$ and $H_C$

3. Integral Representations of $H_B$

**Theorem 2.** Each of the following integral representations for $H_B$ holds true:

$$H_B(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1) \Gamma(a_2)} \int_0^1 \xi^{a_1-1} (1 - \xi)^{a_2-1} \cdot X_4[a_1 + a_2, a_3; c_1, c_2, c_3; x\xi (1 - \xi), y(1 - \xi), z\xi] \, d\xi$$

$$(3.1) \quad \min\{\Re(a_1), \Re(a_2)\} > 0$$

$$H_B(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) = \frac{\Gamma(a_1 + a_2) (\beta - \gamma)^{a_1} (\alpha - \gamma)^{a_2}}{\Gamma(a_1) \Gamma(a_2) (\beta - \alpha)^{a_1 + a_2 - 1}} \cdot \int_0^\beta (\beta - \xi)^{a_2-1} (\xi - \alpha)^{a_1-1} (\xi - \gamma)^{-a_1-a_2} \cdot X_4(a_1 + a_2, a_3; c_1, c_2, c_3; \sigma_1 x, \sigma_2 y, \sigma_3 z) \, d\xi$$

$$(3.2) \quad \min\{\Re(a_1), \Re(a_2)\} > 0; \quad \gamma < \alpha < \beta,$$

where

$$\sigma_1 := \frac{(\alpha - \gamma)(\beta - \gamma)(\xi - \alpha)(\beta - \xi)}{(\beta - \alpha)^2 (\xi - \gamma)^2}; \quad \sigma_2 := \frac{(\alpha - \gamma)(\beta - \xi)}{(\beta - \alpha)(\xi - \gamma)}$$

and

$$\sigma_3 := \frac{(\beta - \gamma)(\xi - \alpha)}{(\beta - \alpha)(\xi - \gamma)};$$

$$(3.3) \quad H_B(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) = 2\frac{\Gamma(a_1 + a_2)}{\Gamma(a_1) \Gamma(a_2)} \cdot \int_0^\frac{\pi}{2} (\sin^2 \xi)^{a_1 - \frac{1}{2}} (\cos^2 \xi)^{a_2 - \frac{1}{2}} X_4(a_1 + a_2, a_3; c_1, c_2, c_3; \sigma_1 x, \sigma_2 y, \sigma_3 z) \, d\xi$$

$$(\min\{\Re(a_1), \Re(a_2)\} > 0),$$

where

$$\sigma_1 := \sin^2 \xi \cos^2 \xi; \quad \sigma_2 := \cos^2 \xi \quad \text{and} \quad \sigma_3 := \sin^2 \xi;$$

$$(3.4) \quad H_B(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) = 2\frac{\Gamma(a_1 + a_2) (1 + \lambda)^{a_1}}{\Gamma(a_1) \Gamma(a_2)} \cdot \int_0^\frac{2\pi}{1 + \lambda \sin^2 \xi} (\sin^2 \xi)^{a_1 - \frac{1}{2}} (\cos^2 \xi)^{a_2 - \frac{1}{2}} X_4(a_1 + a_2, a_3; c_1, c_2, c_3; \sigma_1 x, \sigma_2 y, \sigma_3 z) \, d\xi$$

$$(\min\{\Re(a_1), \Re(a_2)\} > 0; \quad \lambda > -1),$$
where

\[
\sigma_1 := \frac{(1 + \lambda) \sin^2 \xi \cos^2 \xi}{(1 + \lambda \sin^2 \xi)^2}, \quad \sigma_2 := \frac{\cos^2 \xi}{1 + \lambda \sin^2 \xi} \quad \text{and} \quad \sigma_3 := \frac{(1 + \lambda) \sin^2 \xi}{1 + \lambda \sin^2 \xi};
\]

(3.5)

\[
H_B(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) = \frac{2\Gamma(a_1 + a_2) \lambda^{a_1}}{\Gamma(a_1) \Gamma(a_2)} \int_0^\pi \frac{(\sin^2 \xi)^{a_1 - \frac{1}{2}} (\cos^2 \xi)^{a_2 - \frac{1}{2}}}{(\cos^2 \xi + \lambda \sin^2 \xi)^{a_1 + a_2}} X_4(a_1 + a_2, a_3; c_1, c_2, c_3; \sigma_1 x, \sigma_2 y, \sigma_3 z) \, d\xi
\]

(\min\{\Re(a_1), \Re(a_2)\} > 0; \lambda > 0),

where

\[
\sigma_1 := \frac{\lambda \sin^2 \xi \cos^2 \xi}{(\cos^2 \xi + \lambda \sin^2 \xi)^2}, \quad \sigma_2 := \frac{\cos^2 \xi}{\cos^2 \xi + \lambda \sin^2 \xi} \quad \text{and} \quad \sigma_3 := \frac{\lambda \sin^2 \xi}{\cos^2 \xi + \lambda \sin^2 \xi}.
\]

Here \(X_4\) denotes one of Exton’s twenty hypergeometric functions defined by (see [3] and [13, p. 84, Entry (45a)])

\[
X_4(a_1, a_2; c_1, c_2, c_3; x, y, z) = \sum_{m,n,p=0}^\infty \frac{(a_1)_{2m+n+p}(a_2)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m y^n z^p}{m! \ n! \ p!}
\]

(3.6)

\(r := |x|; \ s := |y|; \ t := |z|; \ 2\sqrt{r} + (\sqrt{s} + \sqrt{t})^2 < 1\).

Proof. A similar argument as in the demonstration of Theorem 1 will establish the results asserted by Theorem 2. Indeed, instead of the Gauss hypergeometric series in (2.6), we make use of the double hypergeometric series in (3.6) for Exton’s function \(X_4\).

Alternatively, the assertions (3.1) to (3.5) of Theorem 2 can be proven directly and much more systematically by first writing the definition (1.2) in the following form:

\[
H_B(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z)
= \sum_{m,n,p=0}^\infty \frac{(a_1)_{m+p}(a_2)_{m+n}(a_3)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m y^n z^p}{m! \ n! \ p!}
\]

\[
= \sum_{m,n,p=0}^\infty \frac{(a_1 + a_2)_{2m+n+p}(a_3)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p} \frac{(a_1)_{m+p}(a_2)_{m+n}}{(a_1 + a_2)_{2m+n+p}} \frac{x^m y^n z^p}{m! \ n! \ p!}
\]

\[
= \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1) \Gamma(a_2)} \sum_{m,n,p=0}^\infty \frac{(a_1 + a_2)_{2m+n+p}(a_3)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m y^n z^p}{m! \ n! \ p!}
\]

\(\cdot B(a_1 + m + p, a_2 + m + n),\)
replacing the Beta function:

\[ B(a_1 + m + p, a_2 + m + n) \quad (\text{min}\{\Re(a_1), \Re(a_2)\} > 0) \]

by one or the other of its numerous Eulerian integral representations in (for example) (2.10) to (2.12), and then interpreting the resulting triple hypergeometric series by means of the definition (2.6). In this manner, of course, we can derive a considerably large number of other integral representations for \( H_B \) involving the triple hypergeometric function \( X_4 \) defined by (2.6).

\[ \boxdot \]

4. Integral Representations of \( H_C \)

**Theorem 3.** Each of the following integral representations for \( H_C \) holds true:

\[
H_C(a_1, a_2, a_3; c; x, y, z) = \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(c - a_1)} \int_0^1 \xi^{a_1-1} (1 - \xi)^{c-a_1-1} \\
\cdot (1 - x\xi)^{-a_2} (1 - z\xi)^{-a_3} \text{ } _2F_1(a_2, a_3; c - a_1; \frac{y(1 - \xi)}{(1 - x\xi)(1 - z\xi)}) \, d\xi \\
(\Re(c) > \Re(a_1) > 0);
\]

\[
H_C(a_1, a_2, a_3; c; x, y, z) = \frac{\Gamma(a_1 + a_2 + a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^1 \int_0^1 \xi^{a_1-1}\eta^{a_1+a_2-1} (1 - \xi)^{a_2-1} (1 - \eta)^{a_3-1} \\
\cdot _2F_1\left(\frac{a_1 + a_2 + a_3}{2}, \frac{a_1 + a_2 + a_3 + 1}{2}; c; \Xi(\xi, \eta; x, y, z)\right) \, d\xi \, d\eta \\
(\text{min}\{\Re(a_1), \Re(a_2), \Re(a_3)\} > 0),
\]

where

\[
\Xi(\xi; x, y, z) := 4\eta [x\xi (1 - \xi) + y (1 - \xi) (1 - \eta) + z\xi (1 - \eta)];
\]

\[
H_C(a_1, a_2, a_3; c; x, y, z) = \frac{\Gamma(c)(1 + \lambda)^{a_1}}{\Gamma(a_1)\Gamma(c - a_1)} \int_0^1 \xi^{a_1-1} (1 - \xi)^{c-a_1-1} \\
\cdot (1 + \lambda\xi)^{a_2+a_3-c} [1 + \lambda\xi - (1 + \lambda) x\xi]^{-a_2} [1 + \lambda\xi - (1 + \lambda) z\xi]^{-a_3} \\
\cdot _2F_1\left(a_2, a_3; c - a_1; \frac{y(1 + \lambda\xi)(1 - \xi)}{[1 + \lambda\xi - (1 + \lambda) x\xi][1 + \lambda\xi - (1 + \lambda) z\xi]}\right) \, d\xi \\
(\Re(c) > \Re(a_1) > 0; \lambda > -1);
\]
\[ H_C(a_1, a_2, a_3; c; x, y, z) = \frac{\Gamma(c) (\beta - \gamma)^{a_1} (\alpha - \gamma)^{c-a_1}}{\Gamma(a_1) \Gamma(c-a_1) (\beta - \alpha)^{c-a_2-a_3-1}} \int_0^\beta (\beta - \xi)^{c-a_1-1} \]

\(\cdot (\xi - \alpha)^{a_1-1} (\xi - \gamma)^{a_2+a_3-c} [((\beta - \alpha) (\xi - \gamma) - (\beta - \gamma) (\xi - \alpha) x]^{-a_2}
\]

\(\cdot [((\beta - \alpha) (\xi - \gamma) - (\beta - \gamma) (\xi - \alpha) z]^{-a_3} \quad 2F_1(a_2, a_3; c-a_1; \sigma_y) \, d\xi
\]

where

\[ \sigma := \frac{(\beta - \alpha) (\alpha - \gamma) (\xi - \gamma) (\beta - \xi)}{[(\beta - \alpha) (\xi - \gamma) - (\beta - \gamma) (\xi - \alpha) x] [(\beta - \alpha) (\xi - \gamma) - (\beta - \gamma) (\xi - \alpha) z]};
\]

\[ H_C(a_1, a_2, a_3; c; x, y, z) = \frac{\Gamma(c)}{\Gamma(a_1) \Gamma(c-a_1)} \int_0^\infty \xi^{a_1-1} (1+\xi)^{a_2+a_3-c}
\]

\(\cdot (1+\xi - \xi x)^{-a_2} (1+\xi - \xi z)^{-a_3} \quad 2F_1(a_2, a_3; c-a_1; \sigma_y) \, d\xi
\]

where

\[ \sigma := \frac{(1+\xi)}{(1+\xi - \xi x)(1+\xi - \xi z)};
\]

\[ H_C(a_1, a_2, a_3; c; x, y, z) = \frac{2\Gamma(c)}{\Gamma(a_1) \Gamma(c-a_1)} \int_0^{\frac{\pi}{2}} (\sin^2 \xi)^{a_1-\frac{1}{2}} (\cos^2 \xi)^{c-a_1-\frac{1}{2}}
\]

\(\cdot (1-x \sin^2 \xi)^{-a_2} (1-z \sin^2 \xi)^{-a_3} \quad 2F_1(a_2, a_3; c-a_1; \sigma_y) \, d\xi
\]

where

\[ \sigma := \frac{\cos^2 \xi}{(1-x \sin^2 \xi)(1-z \sin^2 \xi)}.
\]

Here \(2F_1\) denotes the Gauss hypergeometric function given by (2.6).

\textbf{Proof.} Our proof of Theorem 3 is much akin to that of Theorem 1, which we have already presented in a reasonably detailed manner.
5. CONCLUDING REMARKS AND OBSERVATIONS

Integral representations for most of the special functions of mathematical physics and applied mathematics have been investigated in the existing literature. Here we have presented only some illustrative integral representations for each of Srivastava’s functions $H_A$, $H_B$ and $H_C$. A variety of integral representations of $H_A$, $H_B$ and $H_C$, which may be different from those presented here, can also be provided. Furthermore, just as we mentioned in connection with the single- and double-integral representations (2.1) and (1.5) for $H_A$, Srivastava’s double-integral representation (1.9) for $H_C$ can easily be deduced from the assertion (4.1) of Theorem 3 by appealing to the classical result (2.9).

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