INFINITELY MANY SOLUTIONS FOR A CLASS OF DEGENERATE ANISOTROPIC ELLIPTIC PROBLEMS WITH VARIABLE EXPONENT

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Abstract. We study the nonlinear degenerate problem

\[- \sum_{i=1}^{N} \partial_{x_i} a_i (x, \partial_{x_i} u) = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,\]

where \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is a bounded domain with smooth boundary, \( \sum_{i=1}^{N} \partial_{x_i} a_i (x, \partial_{x_i} u) \) is a \( \to p (\cdot) \) - Laplace type operator and the nonlinearity \( f \) is \((P^+ + 1)\) - superlinear at infinity (with \( \bar{p}(x) = (p_1(x), p_2(x), ... p_N(x)) \) and \( P^+_+ = \max_{i \in \{1, ..., N\}} \{ \sup_{x \in \Omega} p_i(x) \} \)). By means of the symmetric mountain-pass theorem of Ambrosetti and Rabinowitz, we establish the existence of a sequence of weak solutions in appropriate anisotropic variable exponent Sobolev spaces.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Our study is conducted in the framework of the anisotropic variable exponent Lebesgue-Sobolev spaces. In the domain of PDEs, characterized by Brézis and Browder [5] as "the major bridge between central issues of applied mathematics and physical sciences on the one hand and the central development of mathematical ideas in active areas of pure mathematics on the other", the theory of anisotropic variable exponent Lebesgue-Sobolev spaces is a bridge itself. Indeed, it is a bridge between the anisotropic Sobolev spaces theory developed by [31, 34, 35, 42, 43] and the variable exponent Sobolev spaces theory developed by [8, 9, 10, 11, 12, 21, 28, 29, 30, 38]. This way, under our dazzled eyes, a delta is born, with new forms of life, or, more exactly, since we refer to a delta of mathematics, with new articles [2, 3, 4, 13, 19, 20, 25, 26, 27]. This state of fact is no surprise, since we know that there are some materials with inhomogeneities for the study of which we cannot use the classical Lebesgue-Sobolev spaces \( L^p \) and \( W^{1,p} \) and we should let the exponent \( p \) to vary instead (the need for the variable exponent spaces is confirmed by the large scale of applications in elastic mechanics [45], in the mathematical modeling...
of non-Newtonian fluids [7, 15, 32, 36, 37, 39, 40, 41, 44] and in image restoration [6]). But what happens when we want to consider materials with inhomogeneities that have different behavior on different space directions? Well, in this case we should work on the anisotropic variable exponent Lebesgue-Sobolev spaces $L^{\vec{p}(\cdot)}$ and $W^{1,\vec{p}(\cdot)}$, where $\vec{p}$ verifies the following condition:

(p) $\vec{p}(x) = (p_1(x), p_2(x), ..., p_N(x))$ and $p_i, \ i \in \{1, ..., N\}$, are continuous functions such that $1 < p_i(x) < N$ and $\sum_{i=1}^{N} 1/\inf_{x} p_i(x) > 1$ for all $x$.

In the context of these spaces that will be carefully described in the next section, we are interested in discussing the following problem:

(1) \[
\begin{cases}
- \sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) = f(x, u) & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary and $a_i, f : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. Let us denote by $A_i, F : \Omega \times \mathbb{R} \to \mathbb{R}$

\[
A_i(x, s) = \int_{0}^{s} a_i(x, t) dt \quad \text{for all } i \in \{1, ..., N\},
\]

respectively

\[
F(x, s) = \int_{0}^{s} f(x, t) dt.
\]

We set $C_+(\Omega) = \{ p \in C(\Omega) : \min_{x \in \Omega} p(x) > 1 \}$ and we denote, for any $p \in C_+(\Omega)$,

\[
p^+ = \sup_{x \in \Omega} p(x) \quad \text{and} \quad p^- = \inf_{x \in \Omega} p(x).
\]

We denote by $\vec{P}^+, \vec{P}^- \in \mathbb{R}^N$ the vectors

\[
\vec{P}^+ = (p_1^+, ..., p_N^+), \quad \vec{P}^- = (p_1^-, ..., p_N^-),
\]

and by $P^+_N, P^+_N, P^-_N \in \mathbb{R}^+$ the following:

\[
P^+_N = \max\{p_1^+, ..., p_N^+\}, \quad P^-_N = \max\{p_1^-, ..., p_N^-\}, \quad P^-_N = \min\{p_1^-, ..., p_N^-\}.
\]

We define $P^*_\infty \in \mathbb{R}^+$ and $P^-_{\infty} \in \mathbb{R}^+$ by

\[
P^*_\infty = \frac{N}{\sum_{i=1}^{N} 1/p_i^- - 1}, \quad P^-_{\infty} = \max\{P^+_N, P^-_N\}.
\]

The goal of this paper is to prove the following theorem.
Theorem 1. Suppose that $\overrightarrow{p}$ verifies (p) and, for all $i \in \{1, \ldots, N\}$, the functions $A_i$, $a_i$, $f$ fulfil the conditions:

(A1) $A_i$ is even in $s$, that is, $A_i(x, -s) = A_i(x, s)$ for all $x \in \Omega$;
(A2) there exists a positive constant $c_{1,i}$ such that $a_i$ satisfies the growth condition

$$|a_i(x, s)| \leq c_{1,i}(1 + |s|^{p_i(x)-1}),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$;
(A3) $a_i$ is strictly monotone, that is,

$$(a_i(x, s) - a_i(x, t))(s - t) > 0,$$

for all $x \in \Omega$ and $s, t \in \mathbb{R}$ with $s \neq t$;
(A4) the following inequalities hold:

$$|s|^{p_i(x)} \leq a_i(x, s)s \leq p_i(x) A_i(x, s),$$

for all $x \in \Omega$ and $s \in \mathbb{R}$;

(f1) $f$ is odd in $s$, that is, $f(x, -s) = -f(x, s)$ for all $x \in \Omega$;
(f2) there exist a positive constant $c_2$ and $q \in C(\overline{\Omega})$ with $1 < p^- < P_+^+ < q^- < q^+ < P_+^*$, such that $f$ satisfies the growth condition

$$|f(x, s)| \leq c_2 |s|^{q(x)-1},$$

for all $x \in \Omega$ and $s \in \mathbb{R}$;
(f3) $f$ verifies the Ambrosetti-Rabinowitz type condition: there exists a constant $\mu > P_+^*$ such that for every $x \in \Omega$

$$0 < \mu F(x, s) \leq sf(x, s), \quad \forall s > 0.$$

Then problem (1) admits an unbounded sequence of weak solutions.

Remark 1. Since $f$ is odd in its second variable $s$, we obtain that $F$ is even in $s$ and the relation described by (f3) remains valid for all $s \in \mathbb{R} \setminus \{0\}$. Moreover, by rewriting condition (f3), we can obtain the existence of a positive constant $c_3$ such that

$$F(x, s) \geq c_3 |s|^\mu, \quad \forall x \in \Omega, \forall s \in \mathbb{R}$$

and we can deduce that $f$ is $(P_+^* - 1)$ - superlinear at infinity:

$$|f(x, s)| \geq c_4 |s|^{\mu-1} \quad \forall x \in \Omega, \forall s \in \mathbb{R},$$

where $c_4$ is a positive constant.
As for the conditions imposed on $A_i$ and $a_i$, obviously they are not randomly chosen. In fact, there are already studies where we can find almost identical conditions and, to give some examples, we indicate [20, 25], or [22, 24, 28] if we are referring to problems of the type
\[
\begin{aligned}
-\text{div}(a(x, \nabla u)) &= f(x, u) \quad \text{for } x \in \Omega, \\
u &= 0 \quad \text{for } x \in \partial \Omega,
\end{aligned}
\]
where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}$ verifies conditions resembling to (A1)-(A4). The preference for conditions (A1)-(A4) may be explained by giving two examples of well known operators that satisfy them:

(1) when choosing $a_i(x, s) = |s|^{p_i(x) - 2}s$ for all $i \in \{1, \ldots, N\}$, we have $A_i(x, s) = \frac{1}{p_i(x)}|s|^{p_i(x)}$ for all $i \in \{1, \ldots, N\}$, and we obtain the anisotropic variable exponent Laplace operator
\[
\sum_{i=1}^{N} \partial_{x_i} \left( |\partial_{x_i} u|^{p_i(x) - 2} \partial_{x_i} u \right);
\]

(2) when choosing $a_i(x, s) = (1 + |s|^2)^{(p_i(x) - 2)/2}s$ for all $i \in \{1, \ldots, N\}$, we have $A_i(x, s) = \frac{1}{p_i(x) - 2}[1 + |s|^2]^{(p_i(x) - 2)/2}$ for all $i \in \{1, \ldots, N\}$, and we obtain the anisotropic variable mean curvature operator
\[
\sum_{i=1}^{N} \partial_{x_i} \left[ \left( 1 + |\partial_{x_i} u|^2 \right)^{(p_i(x) - 2)/2} \partial_{x_i} u \right].
\]

In the light of the above said, we point out that our problem is closely related to the problem discussed in [4],
\[
\begin{aligned}
- \sum_{i=1}^{N} \partial_{x_i} \left( |\partial_{x_i} u|^{p_i(x) - 2} \partial_{x_i} u \right) &= f(x, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying conditions (f1)-(f3) and $p_i$ are continuous functions on $\overline{\Omega}$ such that $2 \leq p_i(x) < N$ for all $x \in \overline{\Omega}$ and $i \in \{1, \ldots, N\}$. Their main theorem also asserts the existence of an unbounded sequence of weak solutions. It is clear that our work extends this result since we can consider (2) to be a particular case of problem (1).

2. Abstract Framework

In this section we recall the definition and some important properties of the Lebesgue-Sobolev spaces mentioned above.
Everywhere below we consider \( p, p_i \in C_+ (\overline{\Omega}) \) to be logarithmic Hölder continuous. The variable exponent Lebesgue space is defined by

\[
L^p (\cdot) (\Omega) = \left\{ u : u \text{ is a measurable real–valued function such that } \int_\Omega |u(x)|^{p(x)} \, dx < \infty \right\}
\]

endowed with the Luxemburg norm

\[
|u|_{p(\cdot)} = \inf \left\{ \mu > 0 : \int_\Omega \frac{|u(x)|^{p(x)}}{\mu} \, dx \leq 1 \right\}.
\]

Notice that for \( p \) constant this norm becomes the norm

\[
|u|_p = \left( \int_\Omega |u|^p \right)^{1/p},
\]

that is, the norm of the classical Lebesgue space \( L^p \).

The space \( (L^p (\cdot) (\Omega), \cdot \, |_{p(\cdot)}) \) has many important qualities. We remind that it is a separable and reflexive Banach space ([21, Theorem 2.5, Corollary 2.7]) and the inclusion between spaces generalizes naturally: if \( 0 < |\Omega| < \infty \) and \( p_1, p_2 \) are variable exponents in \( C_+ (\overline{\Omega}) \) such that \( p_1 \leq p_2 \) in \( \Omega \), then the embedding \( L^{p_2 (\cdot)} (\Omega) \hookrightarrow L^{p_1 (\cdot)} (\Omega) \) is continuous ([21, Theorem 2.8]). In addition, the following Hölder-type inequality

\[
\left| \int_\Omega uv \, dx \right| \leq \left( \frac{1}{p} + \frac{1}{p'} \right) |u|_{p(\cdot)} |v|_{p'(\cdot)} \leq 2 |u|_{p(\cdot)} |v|_{p'(\cdot)}
\]

holds true for all \( u \in L^{p(\cdot)} (\Omega) \) and \( v \in L^{p'(\cdot)} (\Omega) \) ([21, Theorem 2.1]), where we denoted by \( L^{p'(\cdot)} (\Omega) \) the conjugate space of \( L^{p(\cdot)} (\Omega) \), obtained by conjugating the exponent pointwise, that is, \( 1/p(x) + 1/p'(x) = 1 \) ([21, Corollary 2.7]).

Also, the function \( \rho_{p(\cdot)} : L^{p(\cdot)} (\Omega) \to \mathbb{R} \),

\[
\rho_{p(\cdot)} (u) = \int_\Omega |u|^{p(x)} \, dx,
\]

which is called the \( p(\cdot) \)-modular of the \( L^{p(\cdot)} (\Omega) \) space, plays a key role in handling these generalized Lebesgue spaces. We present some of its properties (see again [21]): if \( u \in L^{p(\cdot)} (\Omega) \), \( (u_n) \subset L^{p(\cdot)} (\Omega) \) and \( p^+ < \infty \), then,

\[
|u|_{p(\cdot)} < 1 \quad (= 1; > 1) \quad \iff \quad \rho_{p(\cdot)} (u) < 1 \quad (= 1; > 1)
\]

\[
|u|_{p(\cdot)} > 1 \quad \Rightarrow \quad |u|_{p(\cdot)} \leq \rho_{p(\cdot)} (u) \leq |u|_{p(\cdot)}^{p^+}
\]
\begin{align*}
\text{Let us pass now to the definition of the variable exponent Sobolev space } W^{1,p(\cdot)}(\Omega), \\
W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : \partial_x u \in L^{p(\cdot)}(\Omega), \ i \in \{1, 2, \ldots N\} \right\}
\end{align*}

endowed with the norm

\begin{equation}
\|u\| = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)},
\end{equation}

\((W^{1,p(\cdot)}(\Omega), \| \cdot \|)\) is a separable and reflexive Banach space. \(W^{1,p(\cdot)}_0(\Omega)\), the Sobolev space with zero boundary values defined as the closure of \(C_\infty^0(\Omega)\) with respect to the norm \(\| \cdot \|\), occupies an important place in the theory of variable exponent spaces (see [16, 17]). Note that the norms

\begin{align*}
\|u\|_{1,p(\cdot)} &= \|\nabla u\|_{p(\cdot)}, \\
\|u\|_{p(\cdot)} &= \sum_{i=1}^N |\partial_x u|_{p_i(\cdot)}
\end{align*}

are equivalent to (9) in \(W^{1,p(\cdot)}_0(\Omega)\) and \(W^{1,p(\cdot)}(\Omega)\) is also a separable and reflexive Banach space. Moreover, if \(s \in C_+\overline{(\Omega)}\) and \(s(x) < p^*(x)\) for all \(x \in \overline{\Omega}\), where \(p^*(x) = Np(x)/[N - p(x)]\) if \(p(x) < N\) and \(p^*(x) = \infty\) if \(p(x) \geq N\), then the embedding \(W^{1,p(\cdot)}_0(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)\) is compact and continuous.

Finally we can present the anisotropic variable exponent Sobolev space \(W^{1,\overrightarrow{p}(\cdot)}_0(\Omega)\), where \(\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N\) is the vectorial function

\begin{equation}
\overrightarrow{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot)).
\end{equation}

The space \(W^{1,\overrightarrow{p}(\cdot)}_0(\Omega)\) is defined as the closure of \(C_\infty^0(\Omega)\) under the norm

\begin{equation}
\|u\|_{\overrightarrow{p}(\cdot)} = \sum_{i=1}^N |\partial_x u|_{p_i(\cdot)}.
\end{equation}

The space \(W^{1,\overrightarrow{p}(\cdot)}_0(\Omega)\) allows the adequate treatment of the existence of the weak solutions for problem (1) and can be considered a natural generalization of the variable exponent Sobolev space \(W^{1,p(\cdot)}_0(\Omega)\). On the other hand, playing the
previously announced role of "bridge", \( W^{1,\overrightarrow{p}}_0(\Omega) \) can be considered a natural generalization of the classical anisotropic Sobolev space \( W^{1,\overrightarrow{p}}_0(\Omega) \), where \( \overrightarrow{p} \) is the constant vector \((p_1, ..., p_N)\). \( W^{1,\overrightarrow{p}}_0(\Omega) \) endowed with the norm

\[
\|u\|_{1,\overrightarrow{p}} = \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i}
\]

is a reflexive Banach space for any \( \overrightarrow{p} \in \mathbb{R}^N \) with \( p_i > 1 \) for all \( i \in \{1, ..., N\} \). This result can be easily extended to \( W^{1,\overrightarrow{p}}(\Omega) \), see [27]. Another extension was made in what concerns the embedding between \( W^{1,\overrightarrow{p}}(\Omega) \) and \( L^{q(\cdot)}(\Omega) \) [27, Theorem 1]: if \( \Omega \subset \mathbb{R}^N \) (\( N \geq 3 \)) is a bounded domain with smooth boundary, \( \overrightarrow{p} \) verifies (p) and \( q \in C(\overline{\Omega}) \) verifies \( 1 < q(x) < P_{-\infty} \) for all \( x \in \overline{\Omega} \), then the embedding

\[
W^{1,\overrightarrow{p}}_0(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)
\]

is continuous and compact.

3. AUXILIARY RESULTS

We denote \( W^{1,\overrightarrow{p}}_0(\Omega) \) by \( E \) and we underline the fact that we work under the conditions of Theorem 1. We base the proof of Theorem 1 on the critical point theory, that is, we associate to our problem a functional energy whose critical points represent the weak solutions of the problem.

Let us start by giving the definition of the weak solution for problem (1).

**Definition 1.** By a weak solution to problem (1) we understand a function \( u \in E \) such that

\[
\int_{\Omega} \left[ \sum_{i=1}^{N} a_i(x, \partial_{x_i} u) \partial_{x_i} \varphi - f(x,u) \varphi \right] \, dx = 0,
\]

for all \( \varphi \in E \).

The energy functional corresponding to problem (1) is defined as \( I : E \to \mathbb{R} \),

\[
I(u) = \int_{\Omega} \sum_{i=1}^{N} A_i(x, \partial_{x_i} u) \, dx - \int_{\Omega} F(x,u) \, dx.
\]

For all \( i \in \{1, 2, ..., N\} \), we denote by \( J, J_i : E \to \mathbb{R} \) the functionals

\[
J(u) = \int_{\Omega} \sum_{i=1}^{N} A_i(x, \partial_{x_i} u) \, dx \quad \text{and} \quad J_i(u) = \int_{\Omega} A_i(x, \partial_{x_i} u) \, dx.
\]

We recall the following results concerning the functionals \( J_i \).
Lemma 1. ([20, Lemma 3.4]). For $i \in \{1, 2, \ldots N\}$,

(i) the functional $J_i$ is well-defined on $E$;

(ii) the functional $J_i$ is of class $C^1(E, \mathbb{R})$ and

$$\langle J_i'(u), \varphi \rangle = \int_{\Omega} a_i(x, \partial_x u) \partial_x \varphi dx,$$

for all $u, \varphi \in E$.

A simple calculus leads us to the fact that $I$ is well-defined on $E$ and $I \in C^1(E, \mathbb{R})$ with the derivative given by

$$\langle I'(u), \varphi \rangle = \int_{\Omega} \sum_{i=1}^{N} a_i(x, \partial_x u) \partial_x \varphi dx - \int_{\Omega} f(x, u) \varphi dx,$$

for all $u, \varphi \in E$. It is easy to see that the critical points of $I$ are weak solutions to (1). Therefore we are preoccupied with the existence of critical points. A major help is provided by the mini-max principles, see for example [1, 33]. Here we focus on the symmetric mountain-pass theorem of Ambrosetti and Rabinowitz:

Theorem 2. ([18, Theorem 11.5]). Let $X$ be a real infinite dimensional Banach space and $\Phi \in C^1(X; \mathbb{R})$ a functional satisfying the Palais-Smale condition (that is, any sequence $(u_n)_n \subset X$ such that $(\Phi(u_n))_n$ is bounded and $\Phi'(u_n) \rightarrow 0$ admits a convergent subsequence). Assume that $\Phi$ satisfies:

(i) $\Phi(0) = 0$ and there are constants $\rho, r > 0$ such that

$$\Phi|_{\partial B_{\rho}} \geq r,$$

(ii) $\Phi$ is even, and

(iii) for all finite dimensional subspaces $\tilde{X} \subset X$ there exists $R = R(\tilde{X}) > 0$ such that

$$\Phi(u) \leq 0 \text{ for } u \in \tilde{X} \setminus B_R(\tilde{X}).$$

Then $\Phi$ possesses an unbounded sequence of critical values characterized by a mini-max argument.

To adapt the usual variational methods described by [14, 23] so that we can work on the anisotropic variable exponent Sobolev spaces is not an easy task. Especially when we think at the fact that we inherited the variable exponent from the variable exponent spaces and, in a "world" of partial differential equations, to depend on $x$ may be viewed as a serious "crime". In addition, by the legacy received from the anisotropic spaces, we are dealing with more than just one variable exponent since $p(\cdot)$ is a vector having continuous functions as components. Therefore we must transform our techniques in such manner that we can succeed to overcome all the difficulties and to verify the conditions of Theorem 2. In order to do so, we need the following result, too.
Lemma 2. The operator $J'$ is of type $(S_+)$ on $E$, that is, if $(u_n)_n \subset E$ is weakly convergent to $u \in E$ and
\[ \lim_{n \to \infty} \langle J'(u_n), u_n - u \rangle \leq 0, \]
then $(u_n)_n$ converges strongly to $u$ in $E$.

Proof. The idea of the proof is the same as in [24, Theorem 4.1] because our lemma extends this theorem from the case of the $p(\cdot)$ - Laplace type operators to the case of the $\tilde{p}(\cdot)$ - Laplace type operators. Therefore we follow the reasoning from [24] and we use Vitali’s convergence theorem in order to show that
\[ \lim_{n \to \infty} \int_\Omega \sum_{i=1}^N |\partial_x_i u_n - \partial_x_i u|^{p_i(x)} dx = 0. \]
Consequently, we divide our proof into two parts.

Claim 1. The sequence $\left( \sum_{i=1}^N |\partial_x_i u_n - \partial_x_i u|^{p_i(x)} \right)_n$ is uniformly integrable in $\Omega$, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $H$ is a measurable subset of $\Omega$ with $\text{meas}(H) \leq \delta$, where $\text{meas}(H)$ denotes the Lebesgue measure of $H$, then
\[ \int_H \sum_{i=1}^N |\partial_x_i u_n - \partial_x_i u|^{p_i(x)} dx \leq \varepsilon \quad \forall n \in \mathbb{N}. \]
Since for all $n \in \mathbb{N}$ and for all $x \in \Omega$ there exists $c_5 > 0$ such that
\[ \sum_{i=1}^N |\partial_x_i u_n - \partial_x_i u|^{p_i(x)} \leq c_5 \left( \sum_{i=1}^N |\partial_x_i u_n|^{p_i(x)} + \sum_{i=1}^N |\partial_x_i u|^{p_i(x)} \right) \]
and $|\partial_x_i u|^{p_i(x)} \in L^1(\Omega)$, if we prove that the sequence $\left( \sum_{i=1}^N |\partial_x_i u_n|^{p_i(x)} \right)_n$ is uniformly integrable in $\Omega$, then we have the uniform integrability of $\left( \sum_{i=1}^N |\partial_x_i u_n - \partial_x_i u|^{p_i(x)} \right)_n$. Let us show that $\left( \sum_{i=1}^N |\partial_x_i u_n|^{p_i(x)} \right)_n$ is uniformly integrable.

We know that $(u_n)_n$ is weakly convergent to $u$ and we deduce that
\[ \lim_{n \to \infty} \int_\Omega \sum_{i=1}^N a_i(x, \partial_x_i u) (\partial_x_i u_n - \partial_x_i u) \ dx = 0. \]
From this, (10) and (A3) we get
\[ \lim_{n \to \infty} \int_\Omega \sum_{i=1}^N [a_i(x, \partial_x_i u) - a_i(x, \partial_x_i u)] (\partial_x_i u_n - \partial_x_i u) \ dx = 0. \]
The above relation assures us that for any $\varepsilon > 0$ and any measurable subset $H$ of $\Omega$ there exists $N \in \mathbb{N}$ such that
\[
\int_H \sum_{i=1}^{N} [a_i(x, \partial_x u_n) - a_i(x, \partial_x u)] (\partial_x u_n - \partial_x u) \, dx \leq \frac{\varepsilon}{6} \quad \text{for all } n \geq N.
\]
We fix $\varepsilon > 0$. Then there exists $\delta_1 > 0$ such that if $H$ is a measurable subset of $\Omega$ with $\text{meas}(H) \leq \delta_1$,
\[
(13) \quad \int_H \sum_{i=1}^{N} [a_i(x, \partial_x u_n) - a_i(x, \partial_x u)] (\partial_x u_n - \partial_x u) \, dx \leq \frac{\varepsilon}{6} \quad \text{for all } n \in \mathbb{N}.
\]
Using the first inequality of (A4) we obtain that
\[
(14) \quad \int_H \sum_{i=1}^{N} a_i(x, \partial_x u_n) \partial_x u_n \, dx \geq \int_H \sum_{i=1}^{N} |\partial_x u_n|^{p_i(x)} \, dx.
\]
By a variant of Young's inequality we have that, given $a, b$ such that
\[
\tau \leq \frac{\varepsilon}{6}, \quad (15) \quad \int_H \sum_{i=1}^{N} [a_i(x, \partial_x u_n) - a_i(x, \partial_x u)] (\partial_x u_n - \partial_x u) \, dx \leq \frac{\varepsilon}{6} \quad \text{for all } n \in \mathbb{N}.
\]
(see ([24, relation 3.14])). This inequality yields
\[
(16) \quad \int_H \sum_{i=1}^{N} a_i(x, \partial_x u_n) \partial_x u_n \, dx \leq \frac{1}{3} \int_H \sum_{i=1}^{N} |\partial_x u_n|^{p_i(x)} \, dx + C \left( \frac{1}{3} \right) \int_H \sum_{i=1}^{N} |a_i(x, \partial_x u)|^{p_i(x)} \, dx.
\]
By (A2),
\[
(17) \quad \int_H \sum_{i=1}^{N} a_i(x, \partial_x u_n) \partial_x u_n \, dx \leq \overline{C}_1 \int_H \sum_{i=1}^{N} |\partial_x u_n| |\partial_x u| \, dx + \overline{C}_1 \int_H \sum_{i=1}^{N} |\partial_x u_n|^{p_i(x) - 1} |\partial_x u| \, dx,
\]
where $C_1 = \max\{c_{i,1} : i \in \{1, 2, \ldots, N\}\}$ and $\overline{C}_1 = \max\{C_1, 1\}$. Relying again on (15), we arrive at
\[
(18) \quad \int_H \sum_{i=1}^{N} a_i(x, \partial_x u_n) \partial_x u_n \, dx \leq \overline{C}_1 \int_H \sum_{i=1}^{N} |\partial_x u_n| \, dx + \overline{C}_1 C \left( \frac{1}{3\overline{C}_1} \right) \int_H \sum_{i=1}^{N} |\partial_x u_n|^{p_i(x)} \, dx.
\]
Putting together (13), (14), (16) and (17) we obtain

\[
\frac{1}{3} \int_H \sum_{i=1}^N |\partial_x u_n|^{p_i(x)} \, dx \\
\leq \frac{\varepsilon}{6} + C \left( \frac{1}{3} \right) \int_H \sum_{i=1}^N |a_i(x, \partial_x u)|^{p_i'(x)} \, dx \\
+ C_1 \int_H \sum_{i=1}^N |\partial_x u| \, dx + C_1 C \left( \frac{1}{3C_1} \right) \int_H \sum_{i=1}^N |\partial_x u|^{p_i(x)} \, dx \\
+ \int_H \sum_{i=1}^N |a_i(x, \partial_x u)||\partial_x u| \, dx.
\]

Taking into account (A2), functions \(|a_i(x, \partial_x u)|^{p_i'(x)}|\partial_x u|\), \(|\partial_x u|\) and \(|a_i(x, \partial_x u)||\partial_x u|\) belong to \(L^1(\Omega)\). Hence there exists \(0 < \delta \leq \delta_1\) such that if \(\text{meas}(H) \leq \delta\) then

\[
\frac{\varepsilon}{6} \geq C \left( \frac{1}{3} \right) \int_H \sum_{i=1}^N |a_i(x, \partial_x u)|^{p_i'(x)} \, dx + \\
+ C_1 \int_H \sum_{i=1}^N |\partial_x u| \, dx + C_1 C \left( \frac{1}{3C_1} \right) \int_H \sum_{i=1}^N |\partial_x u|^{p_i(x)} \, dx + \\
+ \int_H \sum_{i=1}^N |a_i(x, \partial_x u)||\partial_x u| \, dx.
\]

The combination of the previous two inequalities conducts us to

\[
\frac{1}{3} \int_H \sum_{i=1}^N |\partial_x u_n|^{p_i(x)} \, dx \leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6}
\]

and we can conclude that the sequence \(\left(\sum_{i=1}^N |\partial_x u_n|^{p_i(x)}\right)_n\) is uniformly integrable in \(\Omega\), thus the proof of the first claim is complete.

**Claim 2.** The sequence \(\left(\sum_{i=1}^N |\partial_x u_n - \partial_x u|^{p_i(x)}\right)_n\) converges in measure to 0 on \(\Omega\).

In order to prove the second claim, we need to show that

(18) \[
\lim_{n \to \infty} |\partial_x u_n - \partial_x u| = 0 \quad \text{for a.e. } x \in \Omega.
\]

Relation (12) provides the existence of a subset \(U\) of \(\Omega\) with \(\text{meas}(U) = 0\) such that
\[
\lim_{n \to \infty} \sum_{i=1}^{N} |a_i(x, \partial_x u_n) - a_i(x, \partial_x u)| (\partial_x u_n - \partial_x u) = 0 \quad \forall x \in \Omega \setminus U,
\]
therefore, due to (A3), for all \(i \in \{1, \ldots, N\},\)
\[
(19) \quad \lim_{n \to \infty} |a_i(x, \partial_x u_n) - a_i(x, \partial_x u)| (\partial_x u_n - \partial_x u) = 0 \quad \forall x \in \Omega \setminus U.
\]
For \(i \in \{1, \ldots, N\} \) and \(x \in \Omega \setminus U,\) we deduce by (19) that there exists \(M > 0\) such that, for all \(n \in \mathbb{N},\)
\[
a_i(x, \partial_x u_n) \partial_x u_n \leq M + |a_i(x, \partial_x u_n)||\partial_x u_n| + |a_i(x, \partial_x u)||\partial_x u_n| + |a_i(x, \partial_x u)||\partial_x u|.
\]
Using (A2) and (A4) in the above inequality we produce
\[
|\partial_x u_n|^{p(x)} \leq M + c_1(1 + |\partial_x u_n|^{p(x)-1})|\partial_x u_n| + |a_i(x, \partial_x u)||\partial_x u_n| + |a_i(x, \partial_x u)||\partial_x u|
\]
from where we obtain that the sequence \((\partial_x u_n)_n\) is bounded. Passing to a subsequence, there exists \(\xi = \xi(x)\) in \(\mathbb{R}\) such that
\[
\partial_x u_{n_k} \to \xi \quad \text{when} \quad k \to \infty.
\]
Moreover, \(a_i\) is a Carathéodory function, so
\[
a_i(x, \partial_x u_{n_k}) \to a_i(x, \xi) \quad \text{when} \quad k \to \infty.
\]
Then, replacing the sequence \((\partial_x u_n)_n\) by its subsequence \((\partial_x u_{n_k})_k\) in (19) and passing to the limit, we come to
\[
[a_i(x, \xi) - a_i(x, \partial_x u)] (\xi - \partial_x u) = 0.
\]
By (A3) and the uniqueness of the limit we deduce that
\[
\partial_x u_{n_k} \to \partial_x u \quad \text{when} \quad k \to \infty.
\]
Since the above arguments are valid for any subsequence of \((u_n)_n\), we obtain that
\[
\partial_x u_n \to \partial_x u \quad \text{when} \quad n \to \infty,
\]
hence (18) holds and the proof of the second claim is complete.

Combining the statements of the two claims with Vitali’s convergence theorem we establish that (11) takes place and, using (8), we get the strong convergence of \((u_n)_n\) to \(u\) in \(E\).

The proof of Theorem 1 will follow the steps indicated by Theorem 2 and we rely on Lemma 2 to show that \(I\) satisfies the Palais-Smale condition. Furthermore, note that \(I(0) = 0\) and the fact that \(A_i\) and \(F\) are even in the second variable implies that \(I\) is even. We are now ready to prove our main theorem, under the reservation that the calculus techniques are not completely new and some of the arguments used are similar to some arguments used by [4]. However, for the completeness of the proof, we must include them in our work.
4. PROOF OF THE MAIN RESULT

Keeping in mind the statement of Theorem 2 and the above comments, we arrange the proof into three parts, namely into three claims.

Claim 1. The energy functional $I$ satisfies condition Palais-Smale.

Let $(u_n)_n \subset E$ be a sequence such that

\begin{equation}
\left| I(u_n) \right| < K, \quad \forall n \geq 1,
\end{equation}

where $K$ is a positive constant, and

\begin{equation}
I'(u_n) \to 0 \quad \text{when } n \to \infty.
\end{equation}

To show that $(u_n)_n$ is bounded, we argue by contradiction and we assume that, passing eventually to a subsequence still denoted by $(u_n)_n$,

\begin{equation}
\|u_n\|_p(\cdot) \to \infty \quad \text{when } n \to \infty.
\end{equation}

By relations (20), (21), (22) we have

$$1 + K + \|u_n\|_p(\cdot) \geq I(u_n) - \frac{1}{\mu}(I'(u_n), u_n)$$

$$\geq \sum_{i=1}^{N} \int_{\Omega} \left[ A_i(x, \partial x_i u_n) - \frac{1}{\mu} a_i(x, \partial x_i u_n) \partial x_i u_n \right] dx$$

$$- \int_{\Omega} \left[ F(x, u_n) - \frac{1}{\mu} u_n f(x, u_n) \right] dx,$$

for sufficiently large $n$, where $\mu$ is the constant from (f3). Using (f3) we get

\begin{equation}
1 + K + \|u_n\|_p(\cdot) \geq \sum_{i=1}^{N} \int_{\Omega} \left[ A_i(x, \partial x_i u_n) - \frac{1}{\mu} a_i(x, \partial x_i u_n) \partial x_i u_n \right] dx.
\end{equation}

From (A4),

$$a_i(x, \partial x_i u_n) \partial x_i u_n \leq p_i(x) A_i(x, \partial x_i u_n),$$

for all $x \in \Omega$ and all $i \in \{1, 2, \ldots N\}$, which implies

$$-\frac{1}{\mu} a_i(x, \partial x_i u_n) \partial x_i u_n \geq -\frac{P^+}{\mu} A_i(x, \partial x_i u_n),$$

for all $x \in \Omega$ and all $i \in \{1, 2, \ldots N\}$. Introducing the previous inequality into relation (23) we obtain

$$1 + K + \|u_n\|_p(\cdot) \geq \left( 1 - \frac{P^+}{\mu} \right) \sum_{i=1}^{N} \int_{\Omega} A_i(x, \partial x_i u_n) dx.$$
From (A4) we also have

\begin{equation}
A_i(x, \partial x, u_n) \geq \frac{1}{P^+} |\partial x, u_n|^{p_i(x)} ,
\end{equation}

for all \( x \in \Omega \) and all \( i \in \{1, 2, \ldots, N\} \), thus

\begin{equation}
1 + K + \|u_n\|^{P_i(-)} \geq \left( \frac{1}{P^+} - \frac{1}{\mu} \right) \sum_{i=1}^{N} \int_{\Omega} |\partial x, u_n|^{p_i(x)} \, dx.
\end{equation}

For every \( n \), let us denote by \( \mathcal{I}_{n_1}, \mathcal{I}_{n_2} \) the indices sets

\[ \mathcal{I}_{n_1} = \{ i \in \{1, 2, \ldots, N\} : |\partial x, u_n|_{p_i(\cdot)} \leq 1 \} \]

and

\[ \mathcal{I}_{n_2} = \{ i \in \{1, 2, \ldots, N\} : |\partial x, u_n|_{p_i(\cdot)} > 1 \} . \]

Using (4), (5), (6) and (25) we infer

\begin{align*}
1 + K + \|u_n\|^{P_i(-)} & \geq \left( \frac{1}{P^+} - \frac{1}{\mu} \right) \left( \sum_{i \in \mathcal{I}_{n_1}} |\partial x, u_n|_{p_i(\cdot)}^{P_i^+} + \sum_{i \in \mathcal{I}_{n_2}} |\partial x, u_n|_{p_i(\cdot)}^{P_i^-} \right) \\
& \geq \left( \frac{1}{P^+} - \frac{1}{\mu} \right) \left( \sum_{i=1}^{N} |\partial x, u_n|_{p_i(\cdot)}^{P_i^-} - \sum_{i \in \mathcal{I}_{n_1}} |\partial x, u_n|_{p_i(\cdot)}^{P_i^-} \right) \\
& \geq \left( \frac{1}{P^+} - \frac{1}{\mu} \right) \left( \sum_{i=1}^{N} |\partial x, u_n|_{p_i(\cdot)}^{P_i^-} - N \right) .
\end{align*}

By the generalized mean inequality or the Jensen inequality applied to the convex function \( a : \mathbb{R}^+ \to \mathbb{R}^+ \), \( a(t) = t^{P_i^-} \), \( P_i^- > 1 \) we get

\begin{equation}
1 + K + \|u_n\|^{P_i(-)} \geq \left( \frac{1}{P^+} - \frac{1}{\mu} \right) \left( \frac{\|u_n\|^{P_i^-}_{P_i(\cdot)}}{N^{P_i^- - 1} - N} \right) .
\end{equation}

Dividing by \( \|u_n\|^{P_i^-}_{P_i(\cdot)} \) in the above relation and passing to the limit as \( n \to \infty \) we obtain a contradiction, hence \( (u_n)_n \) is bounded in \( E \). The fact that \( E \) is reflexive yields the existence of a \( u_0 \in E \) such that, up to a subsequence, \( (u_n)_n \) converges weakly to \( u_0 \) in \( E \). It remains to show that \( (u_n)_n \) converges strongly to \( u_0 \) in \( E \).

Since \( q^+ < P^- = P_{-\infty} \), the embedding \( E \hookrightarrow L^{q(\cdot)}(\Omega) \) is compact, which implies that \( (u_n)_n \) converges strongly to \( u_0 \) in \( L^{q(\cdot)}(\Omega) \). By (f2) and the Holder-type inequality (3),

\begin{equation}
\left| \int_{\Omega} f(x, u_n)(u_n - u_0) \, dx \right| \leq 2c_2 \left\| u_n \right\|^{q(x)-1}_{q(\cdot)} \left\| u_n - u_0 \right\|_{q(\cdot)} .
\end{equation}
By (26), (7) and the strong convergence of \((u_n)_n\) to \(u_0\) in \(L^{q^*}(\Omega)\), we deduce
\[
\lim_{n \to \infty} \int_{\Omega} f(x, u_n)(u_n - u_0) \, dx = 0.
\]
By (21) we deduce
\[
\lim_{n \to \infty} \langle I'(u_n), u_n - u_0 \rangle = 0,
\]
that is
\[
\lim_{n \to \infty} \int_{\Omega} N \sum_{i=1}^{N} a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u_0) \, dx = 0.
\]
Joining together (27) and (28), we find that
\[
\lim_{n \to \infty} \int_{\Omega} N \sum_{i=1}^{N} a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u_0) \, dx = 0.
\]
The statement of Lemma 2 completes the proof of the first claim.

**Claim 2.** There exist \(\rho, r > 0\) such that \(I(u) \geq r > 0\), for any \(u \in E\) with \(\|u\|_{\beta(\cdot)} = \rho\).

Note that, by (24),
\[
I(u) \geq \frac{1}{P^+} \int_{\Omega} \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)} dx - \int_{\Omega} F(x, u) dx, \quad \text{for all } u \in E.
\]
For \(\rho < 1\) we consider \(u \in E\) such that \(\|u\|_{\beta(\cdot)} = \rho < 1\). Thus \(|\partial_{x_i} u|_{p_i(\cdot)} < 1\) and, by (6),
\[
\int_{\Omega} \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)} dx \geq \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}^{P_i^+},
\]
for all \(u \in E\) with \(\|u\|_{\beta(\cdot)} < 1\).

Again, by the generalized mean inequality or the Jensen inequality applied to the convex function \(b : \mathbb{R}^+ \to \mathbb{R}^+, b(t) = t^{P_i^+}, P_i^+ > 1\), we come to
\[
\sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}^{P_i^+} \geq N \left( \frac{\sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(\cdot)}}{N} \right)^{P_i^+}.
\]
By (30) and (31) there exists \(k_0 > 0\) such that
(32) \[ \frac{1}{P_+^N} \int_\Omega \sum_{i=1}^N |\partial_x u|^q_{\Omega} \, dx \geq k_0 \|u\|_{\overline{Q}_+^{n}}^{P_+^{+}}, \]

for all \( u \in E \) with \( \|u\|_{\overline{Q}_+^{n}} < 1 \).

Keeping in mind Remark 1 and using (f2) we obtain,

\[ F(x, s) \leq c_2 \int_0^{|s|} |t|^{q(x)-1} \, dt \leq \frac{c_2}{q-}\bigg(|s|^{q-} + |s|^{q+}\bigg), \]

for all \( x \in \Omega \) and \( s \in \mathbb{R} \). Thus

(33) \[ \int_\Omega F(x, u) \, dx \leq \frac{c_2}{q} \bigg(|u|_{q-}^{q-} + |u|_{q+}^{q+}\bigg) \quad \text{for all} \quad u \in E. \]

Since \( E \hookrightarrow L^{q-}(\Omega), \quad E \hookrightarrow L^{q+}(\Omega) \) continuously we have that there exists a positive constant \( k_1 \) such that, using (33),

\[ \int_\Omega F(x, u) \, dx \leq k_1 \|u\|_{\overline{Q}_+^{n}}^{q-} \quad \text{for all} \quad u \in E \text{ with } \|u\|_{\overline{Q}_+^{n}}^{q-} < 1. \]

Combining the above relation with (32) and (29) we have

\[ I(u) \geq k_0 \|u\|_{\overline{Q}_+^{n}}^{P_+^{+}} - k_1 \|u\|_{\overline{Q}_+^{n}}^{q-} \quad \text{for all} \quad u \in E \text{ with } \|u\|_{\overline{Q}_+^{n}}^{q-} < 1, \]

where \( k_0 = \frac{1}{P_+^{+}N^{P_+^{+}-1}} \). Therefore,

\[ I(u) \geq \|u\|_{\overline{Q}_+^{n}}^{P_+^{+}} \left(k_0 - k_1 \|u\|_{\overline{Q}_+^{n}}^{q-} - P_+^{+}\right) \quad \text{for all} \quad u \in E \text{ with } \|u\|_{\overline{Q}_+^{n}}^{q-} < 1. \]

We denote by \( g : [0, 1] \to \mathbb{R} \) the function defined by

\[ g(t) = k_0 - k_1 t^{q-} - P_+^{+} \]

and we point out the fact that \( g \) is positive in a neighborhood of the origin. Since we can choose \( 0 < \rho < 1 \) sufficiently small, the proof of our second claim is complete.

**Claim 3.**

For any finite dimensional subspace \( \tilde{E} \subset E \) there exists \( R = R(\tilde{E}) > 0 \) such that

\[ I(u) \leq 0 \text{ for all } u \in \tilde{E} \setminus B_{R}(\tilde{E}). \]

By conditions (A1) and (A2),

\[ 0 \leq A_1(x, s) \leq c_{1, s} \int_0^{|s|} \left(1 + |t|^{p_1(x)-1}\right) \, dt = c_{1, s} \left(|s|^{\frac{1}{P_1(x)}} + \frac{|s|^{p_1(x)}}{P_1(x)}\right) \quad \text{for all} \quad x \in \Omega, \ s \in \mathbb{R}, \]
and we obtain

(34) \[ 0 \leq J(v) \leq C_1 \int \Omega \sum_{i=1}^{N} |\partial_{x_i} v| \, dx + \frac{C_1}{P_-} \int \Omega \sum_{i=1}^{N} |\partial_{x_i} v|^{p_i(x)} \, dx \quad \text{for all } v \in E. \]

Let \( \tilde{E} \subset E \) be a finite dimensional subspace, \( u \in \tilde{E} \setminus \{0\} \) and \( t > 1 \). Then, by (34),

\[ J(tu) \leq C_1 \int \Omega \sum_{i=1}^{N} |\partial_{x_i} (tu)| \, dx + \frac{C_1}{P_-} \int \Omega \sum_{i=1}^{N} |\partial_{x_i} (tu)|^{p_i(x)} \, dx \]

and by Remark 1 we infer that

\[ I(tu) \leq C_1 t \int \Omega \sum_{i=1}^{N} |\partial_{x_i} u| \, dx + \frac{C_1 t^{P_-^+}}{P_-} \int \Omega \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)} \, dx - c_3 \mu \int \Omega |u|^\mu \, dx \rightarrow -\infty \quad \text{as } t \rightarrow \infty, \]

since \( \mu > P_-^+ > 1 \).

Notice that, for all \( R > 0 \),

\[
\sup_{\|u\|_{P(\cdot)} = R, u \in \tilde{E}} I(u) = \sup_{\|tu\|_{P(\cdot)} = R, tu \in \tilde{E}} I(tu) = \sup_{\|tu\|_{P(\cdot)} = R, u \in \tilde{E}} I(tu)
\]

and combining the above two relations we get

\[
\sup_{\|u\|_{P(\cdot)} = R, u \in \tilde{E}} I(u) \rightarrow -\infty \quad \text{as } R \rightarrow \infty.
\]

Therefore we can choose \( R_0 > 0 \) sufficiently large such that \( \forall R \geq R_0 \) and \( \forall u \in \tilde{E} \) with \( \|u\|_{P(\cdot)} = R \) we have \( I(u) \leq 0 \). Thus

\[ I(u) \leq 0 \quad \text{for all } u \in \tilde{E} \setminus B_{R_0} \]

and the proof of our final claim is complete.

Finally, taking into account the three claims and using the symmetric mountain-pass theorem of Ambrosetti and Rabinowitz we deduce the existence of an unbounded sequence of weak solutions to problem (1).

\[ \blacksquare \]

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