A NOTE ON HEAT KERNELS OF GENERALIZED HERMITE OPERATORS

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Abstract. In this note, the author obtains heat kernels for the generalised Hermite operators $L = -\Delta + \langle Bx, x \rangle$ where $B$ is a (not necessarily symmetry) semi-positive definite matrix.

1. INTRODUCTION

It is well known that the Hermite operators $L = -\frac{d^2}{dx^2} + \lambda^2 x^2$ and $L = -\frac{d^2}{dx^2} - \lambda^2 x^2$ corresponding to harmonic oscillator and anti-harmonic oscillator play an important role in many mathematical and physical problems (cf. [1, 3, 6, 7, 8]). Hence seeking fundamental solutions of such operators becomes a basic and natural problem.

The purpose of this paper is to consider the heat kernels for the generalised operators taking the form $L = -\Delta + \langle Bx, x \rangle$. In particular, one may concern that $B$ is a positive definite or a negative definite matrix. Recently, [5] obtained the heat kernel for $L$ with any $n \times n$ matrix $B$ by using Hamiltonian formalism. The most striking result they obtained is how the geodesics–solution of the Hamiltonian system–behave for different $B$ in terms of the eigenvalues of $B + B^t$. However, the computation is rather complicated as long as the matrix $B + B^t$ has negative eigenvalues, especially when the dimension $n$ is large.

In some special cases, one can get the explicit heat kernel without solving Hamiltonian system. When $B$ is semi-positive definite, a detailed discussion will be presented in Section 2. In Section 3, resorting to the qualitative conclusion in [5], one also reads off an explicit formulae if $B$ is semi-negative definite.

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2. Heat Kernel for $L$ in $\mathbb{R}^n$ ($B \geq 0$)

One may start with the positive definite case. Consider the generalised Hermite operators of the following form

$$L = -\Delta + \langle Bx, x \rangle$$

where $B$ is a $n \times n$ (not necessarily symmetry) positive definite, $\langle \cdot, \cdot \rangle$ denotes the standard inner product in Euclidean space $\mathbb{R}^n$. The Hamiltonian function associated with $L$ is

$$H(\xi, x) = -\langle \xi, \xi \rangle + \langle Bx, x \rangle$$

hence one obtains the corresponding Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial \xi} = -2\xi \quad \text{and} \quad \dot{\xi} = -\frac{\partial H}{\partial x} = -(B + B^t)x$$

The geodesic $x(s)$ between $x_0$ and $x$ in $\mathbb{R}^n$ satisfies the boundary problem

$$\begin{cases}
\dddot{x} = Ax \\
x(0) = x_0, \; x(t) = x
\end{cases}$$

where $A = 2(B + B^t) > 0$. Since $A$ is a symmetry positive definite matrix, one can find an orthogonal matrix $P$ such that $PAP^t = \text{diag} \{ \lambda_1, \ldots, \lambda_n \} =: \Lambda$, where $\lambda_j > 0$ are eigenvalues of $A$. Set

$$y(s) = Px(s), \; y_0 = y(0) = Px(0) = Px_0, \; \text{and} \; y = y(t) = Px(t) = Px,$$

then problem (2.1) is equivalent to

$$\begin{cases}
\dddot{y} = \Lambda y \\
y(0) = y_0, \; y(t) = y
\end{cases}$$

According to [4], the energy function in $y$-variables is

$$E_y = \sum_{j=1}^{n} \lambda_j \left[ y_j^2 + \left( y_0^j \right)^2 - 2y_j y_0^j \cosh \left( t\lambda_j^{1/2} \right) \right] \frac{2 \sinh^2 \left( t\lambda_j^{1/2} \right)}{2}$$

Noticing that $E = \frac{1}{2} (\langle \dot{x}, \dot{x} \rangle - \langle x, \dddot{x} \rangle)$, one obtains

$$E_x = \frac{1}{2} (\langle P\dot{x}, P\dot{x} \rangle - \langle Px, P\dddot{x} \rangle)$$

$$= \frac{1}{2} (\langle \dot{y}, \dot{y} \rangle - \langle y, \dddot{y} \rangle)$$

$$= E_y$$

$$= \sum_{j=1}^{n} \lambda_j \left[ y_j^2 + \left( y_0^j \right)^2 - 2y_j y_0^j \cosh \left( t\lambda_j^{1/2} \right) \right] \frac{2 \sinh^2 \left( t\lambda_j^{1/2} \right)}{2}$$
To move on, one needs some properties on the action function.

**Proposition.** For action function $S = -\int Edt$, the following equalities hold:

\[
|\nabla S|^2 = \langle Ax, x \rangle + 2E_x
\]

(2.3)

\[
\Delta S = \frac{1}{t} tr \left[ \left( tA^{1/2} \right) \coth \left( tA^{1/2} \right) \right]
\]

(2.4)

where $\varphi(M)$ denotes functional calculus of continuous function $\varphi$ on the symmetry positive definite matrix $M$, and $tr(M)$ denotes the trace of matrix $M$.

**Proof.** A direct computation shows that

\[
S = -\int Edt
\]

\[
= \frac{1}{2t} \sum_{j=1}^{n} \left( t\lambda_j^{1/2} \right) \coth \left( t\lambda_j^{1/2} \right) y_j^2 + \frac{1}{2t} \sum_{j=1}^{n} \left( t\lambda_j^{1/2} \right) \coth \left( t\lambda_j^{1/2} \right) (y_j^0)^2
\]

\[
- \frac{1}{t} \sum_{j=1}^{n} \left( t\lambda_j^{1/2} \right) y_j y_j^0
\]

\[
= \frac{1}{2t} \left\langle \sqrt{\left( tA^{1/2} \right) \coth \left( tA^{1/2} \right) y}, \sqrt{\left( tA^{1/2} \right) \coth \left( tA^{1/2} \right) y} \right\rangle
\]

\[
+ \frac{1}{2t} \left\langle \sqrt{\left( tA^{1/2} \right) \coth \left( tA^{1/2} \right) y_0}, \sqrt{\left( tA^{1/2} \right) \coth \left( tA^{1/2} \right) y_0} \right\rangle
\]

\[
- \frac{1}{t} \left\langle \sqrt{\frac{tA^{1/2}}{\sinh \left( tA^{1/2} \right)}} y, \sqrt{\frac{tA^{1/2}}{\sinh \left( tA^{1/2} \right)}} y_0 \right\rangle
\]

\[
= \frac{1}{2t} \left\langle \left( tA^{1/2} \right) \coth \left( tA^{1/2} \right) x, x \right\rangle + \frac{1}{2t} \left\langle \left( tA^{1/2} \right) \coth \left( tA^{1/2} \right) x_0, x_0 \right\rangle
\]

\[
- \frac{1}{t} \left\langle \frac{tA^{1/2}}{\sinh \left( tA^{1/2} \right)} x_0, x \right\rangle
\]

hence,

\[
\partial_j S = \frac{1}{t} \left\langle \left( tA^{1/2} \right) \coth \left( tA^{1/2} \right) x_j, e_j \right\rangle - \frac{1}{t} \left\langle \frac{tA^{1/2}}{\sinh \left( tA^{1/2} \right)} x_0, e_j \right\rangle
\]

\[
|\nabla S|^2 = \sum_{j=1}^{n} (\partial_j S)^2
\]

(2.5)

\[
= \frac{1}{t^2} \left\langle \left( tA^{1/2} \right) \coth \left( tA^{1/2} \right) x, \left( tA^{1/2} \right) \coth \left( tA^{1/2} \right) x \right\rangle
\]

\[
+ \left\langle \frac{tA^{1/2}}{\sinh \left( tA^{1/2} \right)} x_0, \frac{tA^{1/2}}{\sinh \left( tA^{1/2} \right)} x_0 \right\rangle
\]
\[ -2 \left< \left( tA^{1/2} \right) \coth \left( tA^{1/2} \right), \frac{tA^{1/2}}{\sinh (tA^{1/2})}x_0 \right> \]

\[ = \frac{1}{t^2} \left\{ \sum_{j=1}^{n} \left[ \left( t\lambda_j^{1/2} \right) \coth \left( t\lambda_j^{1/2} \right) y_j \right]^2 + \sum_{j=1}^{n} \left( \frac{t\lambda_j^{1/2}}{\sinh (t\lambda_j^{1/2})} y_j^0 \right)^2 \right\} \]

\[ -2 \sum_{j=1}^{n} \frac{t^2\lambda_j \coth (t\lambda_j^{1/2})}{\sinh (t\lambda_j^{1/2})} y_j y_j^0 \]

\[ = \sum_{j=1}^{n} \frac{\lambda_j \cosh^2 (t\lambda_j^{1/2})}{\sinh^2 (t\lambda_j^{1/2})} y_j^2 \]

\[ + \sum_{j=1}^{n} \frac{\lambda_j \left[ y_j^2 + (y_j^0)^2 \right]}{\sinh (t\lambda_j^{1/2})} - \sum_{j=1}^{n} \frac{\lambda_j \left[ y_j^2 + (y_j^0)^2 \right]}{\sinh (t\lambda_j^{1/2})} \]

\[ + \sum_{j=1}^{n} \frac{\lambda_j \cosh (t\lambda_j^{1/2})}{\sinh (t\lambda_j^{1/2})} y_j y_j^0 \]

\[ = \sum_{j=1}^{n} \lambda_j y_j^2 + 2E_x \]

\[ = \left< Ax, x \right> + 2E_x \]

Differentiating equation (2.5) on \( x_j \) once more, one has

\[ \partial^2_{x_j} S = \frac{1}{t} \left< \left( tA^{1/2} \right) \coth \left( tA^{1/2} \right), e_j, e_j \right> \]

Consequently,

\[ \Delta S = \sum_{j=1}^{n} \partial^2_{x_j} S = \frac{1}{t} \text{tr} \left[ \left( tA^{1/2} \right) \coth \left( tA^{1/2} \right) \right] = \sum_{j=1}^{n} \lambda_j^{1/2} \coth \left( t\lambda_j^{1/2} \right) \]

One expects to find the heat kernel of \( L \) in the following form

\[ K \left( x_0, x, t \right) = V \left( t \right) e^{\alpha S} \]

where \( \alpha \) is a real number to be chosen later. Then

\[ (2.6) \quad \partial_t K = e^{\alpha S} \left[ V' \left( t \right) - \alpha EV \left( t \right) \right] \]

and by use of the Proposition

\[ \Delta e^{\alpha S} = \alpha e^{\alpha S} (\alpha |\nabla S|^2 + \Delta S) \]

\[ = \alpha e^{\alpha S} \left[ \alpha \left< Ax, x \right> + 2\alpha E + \sum_{j=1}^{n} \lambda_j^{1/2} \coth \left( t\lambda_j^{1/2} \right) \right] \]
On the one hand, kernel \( K(x_0, x, t) \) of the Heat operator \( P = \partial_t + L = \partial_t - \Delta + \langle Bx, x \rangle \) satisfies \( PK = 0 \) for any \( t > 0 \); on the other hand, noticing that \( \langle Ax, x \rangle = 4 \langle Bx, x \rangle \),

\[
P_k = \partial_t K - V(t) \Delta e^{\alpha S} + \langle Bx, x \rangle V(t) e^{\alpha S}
\]

\[
= K \left[ \frac{V'(t)}{V(t)} - \alpha (1 + 2\alpha) E - \alpha^2 \langle Ax, x \rangle + \langle Bx, x \rangle - \alpha \sum_{j=1}^{n} \lambda_j^{1/2} \coth \left( t\lambda_j^{1/2} \right) \right]
\]

\[
= K \left[ \frac{V'(t)}{V(t)} - \alpha (1 + 2\alpha) E + (1 - 4\alpha^2) \langle Bx, x \rangle - \alpha \sum_{j=1}^{n} \lambda_j^{1/2} \coth \left( t\lambda_j^{1/2} \right) \right].
\]

Let \( \alpha = -\frac{1}{2} \) and \( V(t) \) satisfy the transport equation

\[
\frac{V'(t)}{V(t)} = -\frac{1}{2} \sum_{j=1}^{n} \lambda_j^{1/2} \coth \left( t\lambda_j^{1/2} \right)
\]

Integration yields \( V(t) = \prod_{j=1}^{n} \frac{C_j}{\sinh^{1/2}(\lambda_j^{1/2})} \). Hence kernel \( K \) has the form

\[
K(x_0, x, t) = \left( \prod_{j=1}^{n} \frac{C_j}{\sinh^{1/2}(\lambda_j^{1/2})} \right)
\]

\[
\times e^{-\frac{1}{4t} \left( (tA^{1/2}) \coth(tA^{1/2}) x_0, x_0 \right) - 2 \left( \frac{tA^{1/2}}{\sinh(tA^{1/2})} \right) x_0, x}.
\]

Since kernel \( K \) becomes Gaussian \( \frac{1}{(4\pi t)^{n/2}} e^{-\frac{1}{4t} \|x-x_0\|^2} \) if \( B \to 0 \), one may compare the volume element \( V(t) \) with \( \frac{1}{(4\pi t)^{n/2}} \) to establish the parameters

\[
C_j = \left( \frac{\lambda_j}{16\pi^2} \right)^{1/4}.
\]

**Theorem.** Let \( B \) be a \( n \times n \) positive definite matrix, then \( A = 2 (B + B') \) is a symmetry positive definite matrix whose Jordan normal form is denoted by \( \text{diag}\{\lambda_1, \ldots, \lambda_n\} \) with \( \lambda_j > 0 \). The kernel of the heat operator \( P = \partial_t - \Delta + \langle Bx, x \rangle \) is

\[
K(x_0, x, t) = \frac{1}{(4\pi t)^{n/2}} \left( \prod_{j=1}^{n} \frac{t\lambda_j^{1/2}}{\sinh(t\lambda_j^{1/2})} \right)^{1/2}
\]

\[
\times e^{-\frac{1}{4t} \left( (tA^{1/2}) \coth(tA^{1/2}) x_0, x_0 \right) - 2 \left( \frac{tA^{1/2}}{\sinh(tA^{1/2})} \right) x_0, x}.
\]

(2.8)
Remark. The form of $K$ is still valid if $B \geq 0$, but the terms $\frac{t\lambda_j^{1/2}}{\sinh(t\lambda_j^{1/2})}$, $(tA^{1/2}) \coth(tA^{1/2})$ and $\frac{tA^{1/2}}{\sinh(tA^{1/2})}$ in equation (2.8) should be replaced by their corresponding limiting forms. Specifically, if 

$$PAP^t = \Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_m, 0, \ldots, 0\}$$

with $\lambda_j > 0$, $j = 1, \ldots, m$ and $\lambda_l = 0$, $l = m + 1, \ldots, n$, then the volume element

$$\prod_{j=1}^{n} \frac{t\lambda_j^{1/2}}{\sinh(t\lambda_j^{1/2})} = \prod_{j=1}^{m} \frac{t\lambda_j^{1/2}}{\sinh(t\lambda_j^{1/2})},$$

and

$$\left(\frac{tA^{1/2}}{\sinh(tA^{1/2})}\right) = P^t \text{diag}\left\{\left(\frac{t\lambda_1^{1/2}}{\sinh(t\lambda_1^{1/2})}\right), \ldots, \left(\frac{t\lambda_m^{1/2}}{\sinh(t\lambda_m^{1/2})}\right), 1, \ldots, 1\right\} P,$$

$$\left(\frac{tA^{1/2}}{\coth(tA^{1/2})}\right) = P^t \text{diag}\left\{\left(\frac{t\lambda_1^{1/2}}{\coth(t\lambda_1^{1/2})}\right), \ldots, \left(\frac{t\lambda_m^{1/2}}{\coth(t\lambda_m^{1/2})}\right), 1, \ldots, 1\right\} P.$$

3. OPEN QUESTION

If $B$ has negative eigenvalues, the energy function $E$ is not well defined on singular regions which are of zero measure in $2n + 1$ dimensional Lebesgue measure (cf. [5]). So far the only way to establish the singular regions is to solve the Hamiltonian system, which requires heavy computations for large $n$.

Here one conjectures that the kernel has the following form. In particular, for the same reason mentioned in the Remark above, it is sufficient to formulate the case $B < 0$.

Conjecture. Let $B < 0$ and $A = 2(B + B^t) \sim \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ with $\lambda_j < 0$. Then one kernel of the heat operator $P = \partial_t - \Delta + \langle Bx, x \rangle$ is

$$K(x_0, x, t) = \frac{1}{(4\pi t)^{n/2}} \left(\prod_{j=1}^{n} \frac{t(-\lambda_j)^{1/2}}{\sin\left(\frac{t(-\lambda_j)^{1/2}}{2}\right)}\right)^{1/2} e^{\left\langle \frac{1}{2} \frac{(-A)^{1/2}x_0}{\sinh\left(\frac{t(-A)^{1/2}}{2}\right)} \right\rangle} e^{-\frac{1}{4t}\left[\langle t(-A)^{1/2} \cot(-A)^{1/2} \rangle x_0, x \rangle + \langle t(-A)^{1/2} \cot(-A)^{1/2} \rangle x_0, x_0 \rangle} \right\rangle}$$

a.e. $(x_0, x, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+$. 
REFERENCES


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