HAYMAN’S CONJECTURE IN A $p$-ADIC FIELD

Jacqueline Ojeda

Abstract. In this paper we study the famous Hayman’s conjecture for transcendental meromorphic functions in a $p$-adic field by using methods of $p$-adic analysis and particularly the $p$-adic Nevanlinna theory.

In $\mathbb{C}$, W. K. Hayman’s stated that if $f$ is a transcendental meromorphic function, then $f' + af^m$ has infinitely many zeros that are not zeros of $f$ for each integer $m \geq 3$ and $a \in \mathbb{C} \setminus \{0\}$, which was proved in [2], [6], [8] and [11]. Here we examine the problem in an algebraically closed complete ultrametric field $K$ of characteristic zero. Considering the function $f' + Tf^m$ with $T \in K(x)$, we show that Hayman’s statement holds for each $m \geq 5$ and $m = 1$. Further, if the residue characteristic of $K$ is zero, then the statement holds for each positive integer $m$ different from 2. We also examine the problem inside an “open” disc.

1. INTRODUCTION AND RESULTS

1.1 Definitions, Notations and Main Results

Throughout this paper, $K$ will denote an algebraically closed field of characteristic zero, complete for an ultrametric absolute value. In $K$, the valuation $v$ is defined by a logarithm function $\log: v(x) = -\log|x|$.

We denote by $A(K)$ the set of entire functions in $K$ and by $M(K)$ the set of meromorphic functions in $K$, i.e., the field of fractions of $A(K)$. Obviously, $M(K)$ contains the field $K(x)$ of rational functions. We remember that the elements in $M(K) \setminus K(x)$ are called transcendental functions and have infinitely many zeros or infinitely many poles.

Given $a \in K$ and $r_1, r_2$ such that $0 < r_1 < r_2$, we denote by $\Gamma(a, r_1, r_2)$ the annulus

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\[ \{x \in \mathbb{K} : r_1 < |x - a| < r_2\}, \text{ and given } r > 0, \text{ we denote by } d(a, r^-) \text{ the open disc } \\
\{x \in \mathbb{K} : |x - a| < r\}, \text{ by } C(a, r) \text{ the circle } \{x \in \mathbb{K} : |x - a| = r\}, \text{ and} \\
\text{by } d(a, r) := d(a, r^-) \cup C(a, r) \text{ the closed disc. Consequently, we denote by } \\
\mathcal{A}(d(a, r^-)) \text{ the set of analytic functions in } d(a, r^-), \text{ i.e., the } \mathbb{K}\text{-algebra of power} \\
\text{series } \sum_{n=0} a_n (x - a)^n \text{ converging in } d(a, r^-), \text{ and by } \mathcal{M}(d(a, r^-)) \text{ the set of} \\
\text{meromorphic functions inside } d(a, r^-), \text{ i.e., the field of fractions of } \mathcal{A}(d(a, r^-)). \\
\text{Moreover, we denote by } \mathcal{A}_b(d(a, r^-)) \text{ the } \mathbb{K}\text{-subalgebra of } \mathcal{A}(d(a, r^-)) \text{ consisted of} \\
\text{the bounded analytic functions } f \in \mathcal{A}(d(a, r^-)), \text{ which satisfy } \sup_{n \in \mathbb{N}} |a_n| r^n < \\
+\infty, \text{ and by } \mathcal{M}_b(d(a, r^-)) \text{ the field of fractions of } \mathcal{A}_b(d(a, r^-)). \text{ Finally, we set} \\
\mathcal{A}_u(d(a, r^-)) = \mathcal{A}(d(a, r^-)) \setminus \mathcal{A}_b(d(a, r^-)) \text{ and } \mathcal{M}_u(d(a, r^-)) = \mathcal{M}(d(a, r^-)) \\
\setminus \mathcal{M}_b(d(a, r^-)). \]

The paper aims at studying Hayman’s conjecture for transcendental meromorphic functions, first in a field of any residue characteristic and next in a field of residue characteristic zero. The problem is the following one: let \( f \in \mathcal{M}(\mathbb{K}) \) be 
transcendental and \( T \in \mathbb{K}(x) \). Can we conclude that \( f' + T f^m \) has infinitely many 
zeros that are not zeros of \( f \) ? Setting \( g = \frac{1}{f} \), it is easily seen that the zeros of 
\( f' + T f^m \) which are not zeros of \( f \) are those of \( g' g^{m-2} - T \). Thus, solving Hayman’s conjecture is equivalent to answering the question whether, given \( g \in \mathcal{M}(\mathbb{K}) \) transcendental and \( T \in \mathbb{K}(x) \), \( g' g^n - T \) has infinitely many zeros.

Indeed, let 
\[ g(x) = \frac{1}{f(x)}. \]

Then,
\[
f'(x) + T f^m(x) = \frac{-1}{[g(x)]^2} g'(x) + \frac{T}{[g(x)]^m} \\
= \frac{-1}{[g(x)]^m} (g^{m-2} g'(x) - T),
\]
where we do \( n = m - 2 \).

The question has been studied in complex analysis for many years, considering 
\( T = a \in \mathbb{C} \). In 1959, W. K. Hayman [8] proved that if \( g \) is a transcendental 
meromorphic function, \( a \in \mathbb{C} \setminus \{0\} \) and \( n \geq 3 \), then \( g' g^n - a \) has infinitely many 
zeros. Twenty years later, E. Mues [11] solved the case \( n = 2 \), and finally in 1995 
W. Bergweiler and A. Eremenko [2], and independently H. H. Chen and M. L. Fang [6] proved that this also holds for \( n = 1 \), which completed the proof of Hayman’s conjecture. Thus, in the complex case, we could deduce that \( f' + a f^m \) has infinitely many zeros which are not zeros of \( f \) when \( m \geq 3 \).
Remark 1. In \( \mathbb{C} \), \( f' + f^m \) may have no zero if \( m = 1 \) or \( m = 2 \) as shown by \( f(x) = \exp(x) \) and \( f(x) = \tan(-x) \) respectively.

In \( p \)-adic analysis, we can also obtain results in a similar problem. Before stating the main theorems, we have to recall some notations used in several works in \( p \)-adic analysis, particularly those used by A. Escassut in [7].

Given \( f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}(\mathbb{K}) \) (resp. in \( \mathcal{A}(d(0, R^-)) \)) and \( r > 0 \) (resp. \( r \in ]0, R[ \)), we set

\[
|f|(r) = \lim_{|x| \to r^-, |x| \neq r} |f(x)|.
\]

Indeed, this limit exists and \(|*|\) is an absolute value on \( \mathcal{A}(\mathbb{K}) \) (resp. on \( \mathcal{A}(d(0, R^-)) \)). It has a natural continuation to \( \mathcal{M}(\mathbb{K}) \) (resp. \( \mathcal{M}(d(0, R^-)) \)) by setting \(|f|(r) = \frac{\nu_f(r)}{\nu_h(r)}\) whenever \( f = \frac{g}{h} \), \( g, h \in \mathcal{A}(\mathbb{K}) \) (resp. \( g, h \in \mathcal{A}(d(0, R^-)) \)).

On the other hand, let \( f = \sum_{n \in \mathbb{Z}} a_n x^n \in \mathcal{M}(\mathbb{K}) \) and let \( r > 0 \). Consider \( f \) in the circle \( C(0, r) \). We will denote by \( \nu^+(f, r) \) (resp. \( \nu^-(f, r) \)) the biggest integer \( i \in \mathbb{Z} \) (resp. the smallest integer \( i \in \mathbb{Z} \)) such that \( v(a_i) - i \log r = \inf_{n \in \mathbb{Z}} v(a_n) - n \log r \).

We will only write \( \nu(f, r) \) when \( \nu^+(f, r) = \nu^-(f, r) \).

Remark 2. We now have to recall certain classical properties of meromorphic functions (see Chapter 23 [7]). Let \( f \in \mathcal{M}(d(0, R^-)) \) and let \( r \in ]0, R[ \).

(1) The difference between the number of zeros and that of poles of \( f \) in the circle \( C(0, r) \), taking multiplicities into account, is equal to \( \nu^+(f, r) - \nu^-(f, r) \).

(2) If \( f \) has zeros and poles in the closed disc \( d(0, r') \), and has no zeros and no poles in the annuli \( \Gamma(0, r', r'') \), then \( \nu^+(f, r) = \nu^-(f, r) \) \( \forall r \in ]r', r''[ \).

Throughout the paper we consider \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \), \( R > 1 \) an integer and \( T = \frac{A}{B} \in \mathbb{K}(x) \) with \( A, B \in \mathbb{K}[x] \) having no common zeros.

Theorem 1. Let \( f \in \mathcal{M}(\mathbb{K}) \) be transcendental (resp. \( f \in \mathcal{M}_u(d(0, R^-)) \)). If \( \lim_{r \to +\infty} |T|(r) > 0 \) (resp. \( \lim_{r \to +\infty} |T|(r) > \frac{1}{R} \)), then \( f' + Tf \) has infinitely many zeros that are not zeros of \( f \).

Theorem 2. Let \( f \in \mathcal{M}(\mathbb{K}) \) be transcendental and \( \deg(A) \geq \deg(B) \) (resp. \( f \in \mathcal{M}_u(d(0, R^-)) \)). Let \( m > 2 \) be an integer. If \( \limsup_{r \to +\infty} |f|(r) > 0 \) (resp. \( \limsup_{r \to +\infty} |f|(r) = +\infty \)), then \( f' + Tf^m \) has infinitely many zeros that are not zeros of \( f \).
**Corollary 1.** Let \( f \in \mathcal{M}(\mathbb{K}) \) be transcendental and \( \deg(A) \geq \deg(B) \) (resp. let \( f \in \mathcal{M}_u(d(0, R^-)) \)). If \( f \) has a finite number of poles and \( m > 2 \) is an integer, then \( f' + T f^m \) has infinitely many zeros that are not zeros of \( f \).

**Proof.** Since \( f \) has a finite number of poles and \( f \) is a transcendental meromorphic function in \( \mathbb{K} \) (resp. \( f \in \mathcal{M}_u(d(0, R^-)) \)), then necessarily \( f \) has infinitely many zeros. Therefore, \( \lim_{r \to +\infty} |f(r)| = +\infty \) (resp. \( \lim_{r \to R} |f(r)| = +\infty \)). So, by Theorem 2, we can deduce the corollary.

**Corollary 2.** Let \( g \in \mathcal{M}(\mathbb{K}) \) be transcendental and \( \deg(A) \geq \deg(B) \) (resp. let \( g \in \mathcal{M}_u(d(0, R^-)) \)). If \( g \) has a finite number of zeros, then \( g'g^n - T \) has infinitely many zeros for all \( n \in \mathbb{N}^* \).

**Proof.** Since \( g \) has a finite number of zeros, then \( f = \frac{1}{g} \) has a finite number of poles. So, applying Theorem 2 to \( f \) with \( m \geq 3 \), and considering that \( n = m - 2 \), we can deduce the corollary.

Let \( \hat{\mathbb{K}} \) be an algebraic extension of the field \( \mathbb{K} \). In the following lemma, which is very useful for the proofs of the following theorems, we will denote by \( \hat{d}(0, R^-) \) the open disc \( \{ x \in \hat{\mathbb{K}} : |x| < R \} \) contained in \( \hat{\mathbb{K}} \).

**Lemma 1.** Let \( f \in \mathcal{M}(d(0, R^-)) \) and let \( \hat{f} \) be the meromorphic function defined by \( f \) in \( \hat{d}(0, R^-) \). Then the zeros and the poles of \( \hat{f} \) in \( \hat{d}(0, R^-) \) are exactly the zeros and the poles of \( f \) in \( d(0, R^-) \), taking multiplicities into account.

**Remark 3.** We remember that, given a meromorphic function \( f \) in the open disc \( d(0, R^-) \subset \mathbb{K} \), it is not always possible to find analytic functions \( h, l \) in \( d(0, R^-) \) without common zeros such that \( f = \frac{h}{l} \), except if \( \mathbb{K} \) is spherically complete, i.e., every decreasing filter on \( \mathbb{K} \) has a center in \( \mathbb{K} \) (see Chapter 3 [7] and [10]). In our case, \( \mathbb{K} \) is an algebraically closed complete ultrametric field, therefore it admits a spherically complete algebraically closed extension \( \hat{\mathbb{K}} \) (see Chapter 7 [7]).

Now, in the field \( \mathbb{K} \), consider \( f \in \mathcal{M}(d(0, R^-)) \). It obviously defines a function \( \hat{f} \in \mathcal{M}(\hat{d}(0, R^-)) \) in the field \( \hat{\mathbb{K}} \). And then, we may write \( \hat{f} \) in the form \( \frac{b_0}{l_0} \) with \( b_0, l_0 \in A(\hat{d}(0, R^-)) \) having no common zeros. Moreover, by Lemma 1, all zeros and poles of \( \hat{f} \) in \( \hat{\mathbb{K}} \) actually lie in \( \mathbb{K} \). So, by Theorem 25.5 [7], there exists \( h \in A(d(0, R^-)) \) such that the function \( \hat{h} \in A(\hat{d}(0, R^-)) \) defined in \( \hat{\mathbb{K}} \) satisfies the following:

1. \( h_0 \) divides \( \hat{h} \) in \( A(\hat{d}(0, R^-)) \).
2. The function \( u = \frac{\hat{h}}{\hat{h}_0} \) belongs to \( A_0(\hat{d}(0, R^-)) \).

Then we may set \( l = u l_0 \in A(\hat{d}(0, R^-)) \). Moreover, we check that \( l \) has coefficients in \( \mathbb{K} \) because \( f = \frac{h}{l} \), hence \( l = f h \) belongs to \( \mathcal{M}(d(0, R^-)) \) and has no pole in \( d(0, R^-) \).
In the following theorems, when it is necessary, we shall consider \( f \in \mathcal{M}(\mathbb{K}) \) because clearly \( \mathcal{M}(d(0, R^*)) \subset \mathcal{M}(\mathbb{K}) \).

In the general \( p \)-adic context, the following theorem is the equivalence of this proved by W. K. Hayman (Theorem 9 [8]). In the proofs of this theorem and the following theorems, the previous Remark 3 and Lemma 1 will be useful.

**Theorem 3.** Let \( f \in \mathcal{M}(\mathbb{K}) \) be transcendental and \( \deg(A) \geq \deg(B) \) (resp. \( f \in \mathcal{M}_u(d(0, R^-)) \)). If \( m \geq 5 \) is an integer, then \( f' + Tf^m \) has infinitely many zeros that are not zeros of \( f \). Moreover, \( f' + f^4 \) must have at least one zero in \( \mathbb{K} \) that is not a zero of \( f \).

Considering (1) and the previous theorem, we obtain the following corollaries.

**Corollary 3.** Let \( g \in \mathcal{M}(\mathbb{K}) \) be transcendental and \( \deg(A) \geq \deg(B) \) (resp. \( g \in \mathcal{M}_u(d(0, R^-)) \)). If \( n \geq 3 \) is an integer, then \( g'g^n - T \) has infinitely many zeros.

**Corollary 4.** Let \( g \in \mathcal{M}(\mathbb{K}) \) be transcendental and \( \deg(A) \geq \deg(B) \). Then \( g'g^2 - T \) has at least one zero in \( \mathbb{K} \).

In order to state Theorem 4, we need to recall some classical definitions. Let \( U_{\mathbb{K}} = \{ x \in \mathbb{K} : |x| \leq 1 \} \) and \( W_{\mathbb{K}} = \{ x \in \mathbb{K} : |x| < 1 \} \) be the valuation ring and the valuation ideal of \( \mathbb{K} \) respectively. The residue characteristic of \( \mathbb{K} \) is the characteristic of the quotient of \( U_{\mathbb{K}} \) by \( W_{\mathbb{K}} \) (see Chapter 1 [7]).

**Lemma 2.** Let \( f(x) = \sum_{n=-\infty}^{+\infty} a_n x^n \) be a Laurent series converging for \( r' < |x| < r'' \) and have no zeros and no poles in \( \Gamma(0, r', r'') \). Let \( q = \nu(f, r) \quad \forall r \in ]r', r''[. \) If the residue characteristic of \( \mathbb{K} \) does not divide \( q \), then

\[
|f'(x)| = \frac{|f(x)|}{|x|} \quad \forall x \in \Gamma(0, r', r'').
\]

**Corollary 5.** Let \( f \in \mathcal{M}(d(0, r'')) \). Assume that \( f \) has \( s \) zeros and \( t \) poles in \( d(0, r') \) and has no zeros and no poles in \( \Gamma(0, r', r'') \). If the residue characteristic of \( \mathbb{K} \) does not divide \( s - t \), then \( |f'(x)| = \frac{f(x)}{|x|} \quad \forall x \in \Gamma(0, r', r''). \)

**Proof.** Indeed, by Theorem 23.4 [7], \( \nu^+(f, r) = \nu^-(f, r) \quad \forall r \in ]r', r'\). If we consider \( f = \frac{h}{l} \) with \( h, l \in \mathcal{A}(d(0, r'')) \), we have

\[
\nu(f, r) = \nu(h, r) - \nu(l, r) = s - t,
\]
whenever \( r \in ]r', r''[ \). So, by Lemma 2, we deduce the corollary.

**Definition.** Let \( f \in \mathcal{M}(\mathbb{K}) \) (resp. \( f \in \mathcal{M}(d(0, R^{-})) \)).

A number \( r \in [0, +\infty[ \) (resp. \( r \in ]0, R[ \)) will be said to be \( f \)-suitable if the difference between the number of zeros and that of poles of \( f \) in \( d(0, r) \), taking multiplicities into account, is not a multiple of the residue characteristic of \( \mathbb{K} \).

A sequence \( \{r_n\}_{n \in \mathbb{N}} \subset [0, +\infty[ \) (resp. \( \{r_n\}_{n \in \mathbb{N}} \subset ]0, R[ \)) will be said to be \( f \)-suitable if each \( r_n \) is \( f \)-suitable and \( \lim_{n \to +\infty} r_n = +\infty \) (resp. \( \lim_{n \to +\infty} r_n = R \)).

The function \( f \) will be said to be optimal if there exists a \( f \)-suitable sequence \( \{r_n\}_{n \in \mathbb{N}} \) in \( ]0, +\infty[ \) (resp. in \( ]0, R[ \)).

**Theorem 4.** Let \( g \in \mathcal{M}(\mathbb{K}) \) be transcendental (resp. \( g \in \mathcal{M}_u(d(0, R^{-})) \)) and let \( \{r_n\}_{n \in \mathbb{N}} \) be a \( g \)-suitable sequence. Then \( \frac{d}{g} \) has infinitely many zeros.

**Corollary 6.** Let \( g \in \mathcal{M}(\mathbb{K}) \) be transcendental (resp. \( g \in \mathcal{M}_u(d(0, R^{-})) \)) and let \( \{r_n\}_{n \in \mathbb{N}} \) be a \( g \)-suitable sequence. Then \( g'g^n \) and \( \frac{d}{g} \) have infinitely many zeros whenever \( n \in \mathbb{N}^* \).

**Proof.** Let \( n \in \mathbb{N} \). Observe that \( g'g^n = \left( \frac{d'}{g} \right) g^{n+1} \) and \( \frac{d}{g} = \left( \frac{d'}{g} \right) \frac{1}{g^{n+1}} \). Note that every zero of \( \frac{d'}{g} \) is neither a zero nor a pole of \( g \), every zero and every pole of \( g \) being a simple pole of \( \frac{d'}{g} \). Thereby, since \( \frac{d'}{g} \) has infinitely many zeros, we deduce that \( g'g^n \) and \( \frac{d}{g} \) have infinitely many zeros in \( \mathbb{K} \) (resp. in \( d(0, R^{-}) \)).

**Theorem 5.** Let \( f \in \mathcal{M}(\mathbb{K}) \) be transcendental and optimal. If \( \deg(A) = \deg(B) \) and if \( m \geq 3 \) is an integer, then \( f' + Tf^m \) has infinitely many zeros that are not zeros of \( f \).

Thus, by (1) we may derive the following corollary.

**Corollary 7.** Let \( g \in \mathcal{M}(\mathbb{K}) \) be transcendental and optimal. If \( \deg(A) = \deg(B) \), then \( g'g^n - T \) has infinitely many zeros for every \( n \in \mathbb{N}^* \).

**Theorem 6.** Let \( f \in \mathcal{M}_u(d(0, R^{-})) \) be an optimal function and let \( U = \frac{d}{U} \in \mathcal{M}_u(d(0, R^{-})) \) have the same finite number of zeros and poles in \( d(0, R^{-}) \). If \( m \geq 3 \) is an integer, then \( f' + U f^m \) has infinitely many zeros that are not zeros of \( f \).

By (1), the following corollary is immediate.

**Corollary 8.** Let \( g \in \mathcal{M}_u(d(0, R^{-})) \) be an optimal function and let \( U \in \mathcal{M}_u(d(0, R^{-})) \) have the same finite number of zeros and poles in \( d(0, R^{-}) \). Then \( g'g^n - U \) has infinitely many zeros for every \( n \in \mathbb{N}^* \).
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Since almost every meromorphic function in a field of residue characteristic zero is optimal, by Theorems 5 and 6, we can deduce Corollaries 9 - 10 and 11 - 12 respectively.

**Corollary 9.** Let $f \in \mathcal{M}(K)$ be transcendental and $\deg(A) = \deg(B)$. If $K$ has residue characteristic zero and $m \geq 3$ is an integer, then $f' + Tf^m$ has infinitely many zeros that are not zeros of $f$.

**Corollary 10.** Let $g \in \mathcal{M}(K)$ be transcendental and $\deg(A) = \deg(B)$. If $K$ has residue characteristic zero, then $g'g^n - T$ has infinitely many zeros for every $n \in \mathbb{N}^*$.

**Corollary 11.** Let $f \in \mathcal{M}(d(0, R^-))$ be such that $0$ is neither a zero nor a pole of $f$. Let $r \in ]0, R[$. We denote by $Z(r, f)$ the counting function of zeros of $f$ in $d(0, R^-)$

$$Z(r, f) = \sum_{\omega_\alpha(f) > 0 \atop |\alpha| \leq r} \omega_\alpha(f)(\log r - \log |\alpha|),$$

and similarly, we set

$$\overline{Z}(r, f) = \sum_{\omega_\alpha(f) > 0 \atop |\alpha| \leq r} (\log r - \log |\alpha|).$$
We shall also consider the counting functions of poles of $f$ in $d(0, R^-)$

$$N(r, f) = Z(r, \frac{1}{f}) \quad \text{and} \quad \overline{N}(r, f) = \overline{Z}(r, \frac{1}{f}).$$

The Nevanlinna function $T(r, f)$ is defined by

$$T(r, f) = \max\{Z(r, f) + \log |f(0)|; \ N(r, f)\}.$$ 

A. Boutabaa and A. Escassut in [5], A. Escassut in [7] and P. C. Hu and C. C. Yang in [9] give us results related to the $p$-adic Nevanlinna theory which we will use in the later proofs. Some of them are the followings.

**Lemma 3.** If $f \in \mathcal{A}(K) \setminus K[x]$ (resp. If $f \in \mathcal{A}_u(d(0, R^-))$), then $f$ has infinitely many zeros.

**Lemma 4.** Let $f \in \mathcal{A}(K)$ (resp. Let $f \in \mathcal{A}(d(0, R^-))$) be such that $f(0) \neq 0$ and let $r > 0$ (resp. let $r \in [0, R]$). For any $b \in K$, we have

$$Z(r, f - b) = Z(r, f) + O(1).$$

**Lemma 5.** Let $f \in \mathcal{A}(K)$ (resp. Let $f \in \mathcal{A}(d(0, R^-))$) be such that $f(0) \neq 0$ and let $r > 0$ (resp. let $r \in [0, R]$). The functions $T(r, f)$ and $Z(r, f)$ are equivalent up to an additive constant.

**Proposition 1.** Let $f_i \in \mathcal{M}(K)$ (resp. Let $f_i \in \mathcal{M}(d(0, R^-))$) be such that $f_i(0) \neq 0$, $\infty$ for $i = 1, ..., k$. Then, for $r > 0$ (resp. for $r \in [0, R]$), we have

$$Z(r, \prod_{i=1}^{k} f_i) \leq \sum_{i=1}^{k} Z(r, f_i),$$

$$T(r, \sum_{i=1}^{k} f_i) \leq \sum_{i=1}^{k} T(r, f_i), \quad T(r, \prod_{i=1}^{k} f_i) \leq \sum_{i=1}^{k} T(r, f_i),$$

and $T(r, f)$ is an increasing function of $r$.

As a corollary of Lemma 2.1 [5], considering the previous notations, we obtain the following Lemma 6 that also is known as the version $p$-adic of Jensen’s formula.

**Lemma 6.** Let $f \in \mathcal{M}(K)$ (resp. Let $f \in \mathcal{M}(d(0, R^-))$) be such that $0$ is neither a zero nor a pole of $f$. Then,

$$\log |f|(r) = Z(r, f) - N(r, f) + \log |f(0)|.$$
Proposition 2. Let \( f \in \mathcal{M}(d(0, R^-)) \) be such that \( f(0) \neq 0, \infty \). Then, \( f \in \mathcal{M}_b(d(0, R^-)) \) if and only if \( T(r, f) \) is bounded in \([0, R[\). Let \( f \in \mathcal{M}(d(0, R^-)) \) be such that 0 is neither a zero nor a pole of \( f' \) and let \( S \) be a finite subset of \( \mathbb{K} \). We denote by \( Z_0^S(r, f') \) the counting function of zeros of \( f' \) in \( d(0, r) \) which are not zeros of any \( f - s \) for \( s \in S \). Then,

\[
Z_0^S(r, f') = \sum_{s \in S, w_{\alpha}(f-s)=0, |\alpha| \leq r} w_{\alpha}(f')(\log r - \log |\alpha|).
\]

Now we can state the ultrametric Nevanlinna Second Main Theorem in a basic form.

Theorem N. Let \( \beta_1, ..., \beta_n \in \mathbb{K} \) with \( n \geq 2 \), and let \( f \in \mathcal{M} (\mathbb{K}) \) (resp. let \( f \in \mathcal{M}(d(0, R^-)) \)). Let \( S = \{\beta_1, ..., \beta_n\} \). Assume that none of \( f, f' \) and \( f - \beta_j \) with \( 1 \leq j \leq n \) equals 0 or \( \infty \) at the origin. Then, for all \( r > 0 \) (resp. for all \( r \in [0, R[\)), we have

\[
(n - 1)T(r, f) \leq \sum_{j=1}^{n} Z(r, f - \beta_j) + \mathcal{N}(r, f) - Z_0^S(r, f') - \log r + O(1).
\]

In order to go on, we remember the interesting corollary of the Nevanlinna Second Main Theorem on three small functions for \( p \)-adic analytic functions (see Theorem 4 [12]), which we will use later in the proof of Theorem 3.

Theorem T. Let \( f \in \mathcal{A}(\mathbb{K}) \) (resp. let \( f \in \mathcal{A}(d(0, R^-)) \)) be non-constant such that \( f(0) \neq 0, \infty \), and let \( u_1, u_2 \in \mathcal{A}(\mathbb{K}) \) (resp. let \( u_1, u_2 \in \mathcal{A}(d(0, R^-)) \)) be small functions with respect to \( f \) and not zero at 0. Then,

\[
T(r, f) \leq \mathcal{Z}(r, f - u_1) + \mathcal{Z}(r, f - u_2) + S(r)
\]

where \( S(r) = 2T(r, u_1) + 3T(r, u_2) - \log r + O(1) \).

2. Proofs of the Main Lemmas and Theorems

2.1. Proof of Lemma 1

Proof. It is sufficient to show the claim whenever \( f \in \mathcal{A}(d(0, R^-)) \). Let \( f(x) = \sum_{i=0}^{+\infty} c_i x^i \). Clearly we can notice that every zero of \( f \) in \( d(0, R^-) \) is also a zero of \( \hat{f} \) in \( \hat{d}(0, R^-) \).
Let \( r \in ]0, R[ \) and let \( \alpha_1, \ldots, \alpha_q \) be the zeros of \( f \) in the circle \( C(0, r) \) with
\[
\omega_{\alpha_i}(f) = s_i \quad \text{for} \quad i = 1, \ldots, q.
\]
Thereby, \( f \) is factorized in the form
\[
f = \prod_{i=1}^{q} (x - \alpha_i)^{s_i} g,
\]
where \( g \in \mathcal{A}(d(0, R^-)) \) and \( g(\alpha_i) \neq 0 \) for \( i = 1, \ldots, q \). Observe that this factorization also holds in \( \mathcal{M}(d(0, R^-)) \). Hence \( \alpha_i \) is also a zero of order \( s_i \) of \( \hat{f} \) for \( i = 1, \ldots, q \). Now, suppose that \( \hat{f} \) admits other zeros \( \alpha_{q+1}, \ldots, \alpha_t \) with \( \omega_{\alpha_i}(\hat{f}) = s_i \) for \( i = q + 1, \ldots, t \). By Theorem 23.1 [7], for all \( r \in ]0, R[ \), we have
\[
\nu^+(f, r) - \nu^-(f, r) = \sum_{i=1}^{q} s_i,
\]
and similarly, we have
\[
\nu^+(\hat{f}, r) - \nu^-(\hat{f}, r) = \sum_{i=1}^{t} s_i.
\]

But, we know that \( \nu^+(f, r), \nu^-(f, r), \nu^+(\hat{f}, r), \nu^-(\hat{f}, r) \) are only defined by the coefficients of \( f \). So, for \( r \in ]0, R[ \), we have \( \nu^+(f, r) = \nu^+(\hat{f}, r) \) and \( \nu^-(f, r) = \nu^-(\hat{f}, r) \). Consequently \( t = q \), which finishes the proof.

### 2.2. Proof of Lemma 2

**Proof.** Since \( f \) has no zeros in \( \Gamma(0, r', r'') \), then by Theorem 23.4 [7],
\[
\nu^+(f, r) = \nu^-(f, r) \quad \forall r \in ]r', r''[.
\]
Moreover, since \( q = \nu(f, r) \quad \forall r \in ]r', r''[ \), we have
\[
|f(x)| = |a_q||x|^q \quad \forall x \in \Gamma(0, r', r'')
\]
with \( |a_q||x|^q > |a_n||x|^n \quad \forall q \neq n. \) Consequently, since \( |q| = 1 \) by our assumption that the residue characteristic of \( \mathbb{K} \) does not divide \( q \), we have
\[
|f'(x)| = \left| \sum_{n=-\infty}^{+\infty} na_n x^{n-1} \right| = |a_q||x|^q - 1 = \frac{1}{|x|} |a_q||x|^q.
\]
Therefore, we may deduce that \( |f'(x)| = \frac{|f(x)|}{|x|}. \)

### 2.3. Proof of Theorem 1

**Proof.** Let \( r > 0 \) (resp. \( r \in [1, R[ \)). By Lemma 4 [3], we know that
\[
|f'|(r) \leq \frac{1}{2} |f|(r).
\]
We shall check that there exists a \( \rho \in ]0, +\infty[ \) (resp. \( \rho \in [1, R[ \)) such that
\[
|f'|(r) < |Tf|(r) \quad \forall r \in ]\rho, +\infty[ \) (resp. \( \forall r \in ]\rho, R[ \)). Indeed, if \( f \in \mathcal{M}(\mathbb{K}) \) the existence of \( \rho \) is immediate because \( \lim_{r \to +\infty} |T|(r) > 0. \) Now, suppose that
Suppose first that \( f \) has a finite number of poles. Then, \( f \) has infinitely many zeros in \( \mathbb{K} \) (resp. in \( d(0, R^{-}) \)) because \( f \) is transcendental in \( \mathbb{K} \) (resp. is unbounded in \( d(0, R^{-}) \)). Moreover, there exists an increasing sequence \( \{r_n\}_{n \in \mathbb{N}} \) with \( \lim_{n \to +\infty} r_n = +\infty \) (resp. \( \lim_{n \to +\infty} r_n = R \)), such that \( f \) admits zeros and no poles in \( C(0, r_n) \), such that \( T \) has no zeros and no poles in \( C(0, r_n) \) and such that

\[
|f' + Tf|(r) = |Tf|(r) \quad \forall r \geq r_1.
\]

Since \( |f' + Tf|(r) = |Tf|(r) \) in a neighborhood of \( r_n \), we have

\[
\nu^+(f' + Tf, r_n) - \nu^-(f' + Tf, r_n) = \nu^+(f, r_n) - \nu^-(f, r_n),
\]

where \( \nu^+(f, r_n) - \nu^-(f, r_n) \) is the number of zeros of \( f \) in \( C(0, r_n) \) and \( \nu^+(f' + Tf, r_n) - \nu^-(f' + Tf, r_n) \) is the number of zeros of \( f' + Tf \) in \( C(0, r_n) \) (counting multiplicities). Hence, we may deduce that \( f' + Tf \) has zeros in \( C(0, r_n) \) and the number of zeros of \( f' + Tf \) is equal to the number of zeros of \( f \) in \( C(0, r_n) \) (counting multiplicities).

On the other hand, since each zero of \( f \) in \( C(0, r_n) \) either is not a zero of \( f' + Tf \) or is a zero of \( f' + Tf \) of order strictly lower than its order as a zero of \( f \), by (2) there does exist at least a zero of \( f' + Tf \) that is not a zero of \( f \) in \( C(0, r_n) \). Since this is true for all \( n \in \mathbb{N} \), we obtain that \( f' + Tf \) has infinitely many zeros in \( \mathbb{K} \) (resp. in \( d(0, R^{-}) \)) that are not zeros of \( f \).

Now, suppose that \( f \) has infinitely many poles. Then, there exists an increasing sequence \( \{r_n\}_{n \in \mathbb{N}} \) with \( \lim_{n \to +\infty} r_n = +\infty \) (resp. \( \lim_{n \to +\infty} r_n = R \)), such that \( f \) admits poles in \( C(0, r_n) \), such that \( T \) has no zeros and no poles in \( C(0, r_n) \) and such that

\[
|f' + Tf|(r) = |Tf|(r) \quad \forall r \geq r_1.
\]

Let \( n \in \mathbb{N} \). Let \( s_n \) and \( t_n \) be the number of zeros and that of poles of \( f \) in \( C(0, r_n) \) respectively, and let \( \gamma_n \) and \( \tau_n \) be the number of zeros and that of poles of \( f' + Tf \) in \( C(0, r_n) \) respectively. Then, we deduce that

\[
\nu^+(f, r_n) - \nu^-(f, r_n) = s_n - t_n \quad \text{and} \quad \nu^+(f' + Tf, r_n) - \nu^-(f' + Tf, r_n) = \gamma_n - \tau_n.
\]

Since \( |f' + Tf|(r) = |Tf|(r) \) in a neighborhood of \( r_n \), we have again

\[
\nu^+(f, r_n) - \nu^-(f, r_n) = \nu^+(f' + Tf, r_n) - \nu^-(f' + Tf, r_n).
\]
Consequently, $\gamma_n - \tau_n = s_n - t_n$ in $C(0, r_n)$. But we may observe that $\tau_n$ is the number of poles of $f'$ in $C(0, r_n)$ (counting multiplicities). So, since $T$ has no zeros and no poles in $C(0, r_n)$, we have $\tau_n > t_n$ which implies that $\gamma_n > s_n$. Thus, $f' + Tf$ must have at least one zero in $C(0, r_n)$ that is not a zero of $f$. Since this is true for all $n \in N$, we deduce that $f' + Tf$ has infinitely many zeros in $\mathbb{K}$ (resp. in $d(0, R^-)$) which are not zeros of $f$.

2.4. Proof of Theorem 2

Proof. Assume, without loss of generality, that 0 is neither a zero nor a pole $T f_m$ and $f' + T f_m$. We shall prove that $f$ has infinitely many zeros in $\mathbb{K}$ (resp. in $d(0, R^-)$). First we suppose $f \in \mathcal{M}(\mathbb{K})$. By hypothesis $\limsup_{r \to +\infty} |f|(r) > 0$, i.e., there exist a sequence $\{\Gamma(0, r_n', r_n'')\} \in \mathbb{N}$ with $\lim_{n \to +\infty} r_n'' = +\infty$, and a constant $C > 0$, such that $Z(r, f) \geq N(r, f) + C$  $\forall r \in \bigcup_{n \in \mathbb{N}} [r_n', r_n'']$. If $f$ has a finite number of zeros, say, $q$, then $Z(r, f) = q \log r$ and so $N(r, f) + C \leq q \log r$. Consequently $f$ has a finite number of poles, a contradiction because $f$ is transcendental.

Now, suppose $f \in \mathcal{M}_d(d(0, R^-))$. If $f$ has a finite number of zeros in $d(0, R^-)$, then $\lim_{r \to R^-} Z(r, f) < +\infty$ and hence $\limsup_{r \to R^-} |f|(r) < +\infty$, a contradiction to our hypothesis.

Suppose that the set of zeros of $f' + T f_m$ which are not zeros of $f$ is finite. Then, there exists a $\rho > 0$ (resp. $\rho \in [1, R]$) such that $f' + T f_m$ has no zeros other than the multiple zeros of $f$ in $\mathbb{K} \setminus d(0, \rho)$ (resp. in $\Gamma(0, \rho, R)$) and such that $T$ has no zeros and no poles in $\mathbb{K} \setminus d(0, \rho)$ (resp. in $\Gamma(0, \rho, R)$). So, each pole of $f' + T f_m$ is a pole of $f_m$ of the same multiplicity. Hence,

$$\begin{align*}
N(r, f' + T f_m) - N(\rho, f' + T f_m) \\
= N(r, f_m) - N(\rho, f_m) \quad \forall r \in \mathbb{K} \setminus d(0, \rho) \quad (\text{resp. } \forall r \in ]\rho, R[).
\end{align*}$$

Let $\sigma > \rho$ be such that $C(0, \sigma)$ contains at least one zero of $f$. Each zero of $f$, say, of order $q$, either is not a zero of $f' + T f_m$ or is a zero of $f' + T f_m$ with order $q - 1$. Since $f' + T f_m$ has no zeros in $C(0, r)$ other than the zeros of $f$ and $T$ has no zeros and no poles in $C(0, r)$, clearly the number of zeros of $f' + T f_m$ in $C(0, r)$ (counting multiplicities) is strictly inferior to the number of zeros of $T f_m$ (counting multiplicities). So, the function

$$\Psi(r) = Z(r, f_m) - Z(\rho, f_m) - \left[ Z(r, f' + T f_m) - Z(\rho, f' + T f_m) \right]$$

is strictly increasing in $[\sigma, +\infty)$ (resp. in $[\sigma, R]$).
Now, we will show that there exists an increasing sequence of intervals \([r'_n, r''_n]\) with
\[
\rho < r'_n < r''_n < r'_{n+1}\] and \(\lim_{n \to +\infty} r'_n = +\infty\) (resp. \(\lim_{n \to +\infty} r''_n = R\)), such that
\[
|f' + Tf^m|(r) = |Tf^m|(r) \quad \forall r \in [r'_n, r''_n].
\]
Suppose first that \(f \in \mathcal{M}(\mathbb{K})\). Since
\[
\limsup_{r \to +\infty} |f(r)| > 0,
\]
there exist a sequence of annuli \(\{\Gamma(0, r'_n, r''_n)\}_{n \in \mathbb{N}}\) with \(\rho < r'_n < r''_n\) and \(\lim_{n \to +\infty} r''_n = +\infty\), and a constant \(C > 0\) such that
\[
|f|(r) > C \quad \forall r \in [r'_n, r''_n] \quad \forall n \in \mathbb{N}.
\]
Since \(T\) has no zeros and no poles in \([r'_n, r''_n]\) and \(\deg(A) \geq \deg(B)\), then there
exists a constant \(\lambda > 0\) such that \(|T|(r) \geq \lambda \quad \forall r \in [r'_n, r''_n]\). So
\[
|Tf^m|(r) > C^m \lambda \quad \forall r \in [r'_n, r''_n] \quad \forall n \in \mathbb{N}.
\]
On the other hand, by Lemma 4 [3], \(|f'|(r) \leq \frac{1}{\lambda} |f|(r)\). So, if we consider the
previous observation, we can deduce that
\[
\frac{|f'|}{|Tf^m|}(r) \leq \frac{1}{\lambda} \frac{1}{|Tf^m|(r)} < \frac{1}{\lambda r} \left(\frac{1}{C}\right)^{m-1}.
\]
However, for \(r\) sufficiently large, we have \(\frac{1}{\lambda r} \left(\frac{1}{C}\right)^{m-1} < 1\). Hence \(|f'|(r) < |Tf^m|(r)\).

Thereby,
\[
|f' + Tf^m|(r) = |Tf^m|(r).
\]
Thus, this equality holds in all annulus \(\Gamma(0, r'_n, r''_n)\) when \(r'_n\) is sufficiently large. Consequently, without loss of generality, we may assume that \(|f' + Tf^m|(r) = |Tf^m|(r) \quad \forall r \in [r'_n, r''_n] \quad \forall n \in \mathbb{N}\).

Now, we suppose that \(f \in \mathcal{M}_u(d(0, R^-))\). Since \(\limsup_{r \to R^-} |f|(r) = +\infty\)
there exists a sequence of annuli \(\{\Gamma(0, r'_n, r''_n)\}_{n \in \mathbb{N}}\) with \(\rho < r'_n < r''_n\) and
\(\lim_{n \to +\infty} r''_n = R\), such that \(|f|(r) \geq n \forall r \in [r'_n, r''_n]\) and \(n \in \mathbb{N}\). Since \(T \in \mathbb{K}(x)\), there
exists a constant \(\lambda > 0\) such that \(\inf_{r \in [1, R]} |T|(r) = \lambda\). Then, \(|Tf^m|(r) \geq \lambda |f|(r)^{m-1}\)
\(\forall r \in [r'_n, r''_n] \) and \(n \in \mathbb{N}\). Moreover, we can see that \(|f'|(r) < |f|(r) \forall r \in [r'_n, r''_n]\) because
\(r'_n > 1\). Consequently, when \(n\) is sufficiently large, we have
\[
|f'|(r) < \lambda |f|(r) \leq \lambda n^{m-1} |f|(r) \leq |Tf^m|(r) \quad \forall r \in [r'_n, r''_n],
\]
which implies that \(|f' + Tf^m|(r) = |Tf^m|(r) \forall r \in [r'_n, r''_n] |\).

Therefore, by Lemma 6, we obtain
\[
Z(r, Tf^m + f') - N(r, Tf^m + f')
= Z(r, f^m) - N(r, f^m) + \chi \quad \forall r \in [r'_n, r''_n],
\]
where \(\chi\) is defined as \(m \log |f(0)| - \log |T(0)f^m(0) + f'(0)|\). And by (3) and (4),
we can check that
\[
\Psi(r) = Z(\rho, f' + Tf^m) - N(\rho, f' + Tf^m) - Z(\rho, f^m) - N(\rho, f^m) + \chi.
\]
Consequently $\Psi$ is constant in $[\sigma, +\infty[$ (resp. in $[\sigma, R]$), a contradiction because we have showed that it is strictly increasing. ■

2.5. Proof of Theorem 3

Proof. In order to prove Theorem 3, thanks to Lemma 1, we can place ourselves in $\hat{d}(0, R^-) \subset \mathbb{K}$ in the case when $f \in \mathcal{M}_u(\hat{d}(0, R^-))$. Since $f$ is a transcendental meromorphic function in $\mathbb{K}$ (resp. unbounded in $\hat{d}(0, R^-)$), there exist entire functions $h$, $l \in \mathcal{A}(\mathbb{K})$ (resp. $h$, $l \in \mathcal{A}(\hat{d}(0, R^-))$) without common zeros and at least one of them being transcendental (resp. unbounded) such that $f = \frac{1}{h}$. We can write $h$ in the form $\overline{h}$, where the zeros of $\overline{h}$ are exactly the different zeros of $h$ but all with multiplicity 1. Then, necessarily, $h'$ is multiple of $\overline{h}$ in $\mathcal{A}(\mathbb{K})$ (resp. in $\mathcal{A}(\hat{d}(0, R^-))$). So $h' = u \overline{h}$ with $u \in \mathcal{A}(\mathbb{K})$ (resp. $u \in \mathcal{A}(\hat{d}(0, R^-))$).

Suppose that $f' + T f^m$ has a finite number of zeros in $\mathbb{K}$ (resp. in $\hat{d}(0, R^-)$) which are not zeros of $f$. Then, there exists a polynomial $P \in \mathbb{K}[x]$ of degree $q$, having no common zeros with $Bl$, such that

$$f' + T f^m = \frac{P \overline{h}}{Bl^m}.$$ 

This implies

$$f' f^m = \frac{P \overline{h} - Ah^m}{Bh^m} = \frac{P - A\overline{h}h^{m-1}}{B \overline{h} h^{m-1}}. \tag{5}$$

On the other hand, we note that

$$f' f^m = \frac{t^{m-2}(h' l - hl')}{h^m} = \frac{t^{m-2}(ul - \overline{h}l')}{\overline{h} h^{m-1}}. \tag{6}$$

So, by (5) and (6),

$$Bl^{m-2}(ul - \overline{h}l') = P - A\overline{h}h^{m-1}.$$ 

Let $F = Bl^{m-2}(ul - \overline{h}l')$ and $s = \deg(A)$. Let $r > 0$ (resp. $r \in [1, R]$).

Applying Theorem $T$ to $F$, and noting that $\overline{Z}(r, h) = \overline{Z}(r, \overline{h})^{m-1} = Z(r, \overline{h})$, we obtain

$$T(r, F) \leq \overline{Z}(r, F) + \overline{Z}(r, F - P) + 3T(r, P) - \log r + O(1)$$

$$\leq \overline{Z}(r, B) + \overline{Z}(r, t^{m-2}) + \overline{Z}(r, ul - \overline{h}l')$$

$$+ \overline{Z}(r, A) + \overline{Z}(r, h) + (3q - 1) \log r + O(1)$$

$$\leq Z(r, B) + Z(r, l)$$

$$+ Z(r, ul - \overline{h}l') + Z(r, h) + (3q + s - 1) \log r + O(1). \tag{7}$$
Moreover, we have

\[
T(r, F) = T(r, B) + T(r, t^{m-2}) + T(r, u l - \overline{\theta} l') + O(1)
\]

\[
= Z(r, B) + (m-2)Z(r, l) + Z(r, u l - \overline{\theta} l') + O(1).
\]

Let \(d = 3q + s - 1\). By (7) and (8), we deduce that

\[
(m-3)Z(r, l) \leq Z(r, h) + d \log r + O(1).
\]

Since we assume that the set of zeros of \(f' + T f^m\) that are not zeros of \(f\) is finite, by Theorem 2, we can restrict ourselves to the assumption \(\lim sup f(r) = 0\) (resp. \(\lim sup f(r) < +\infty\)) and therefore \(\lim sup[Z(r, l) - Z(r, h)] = +\infty\) (resp. \(\lim sup[Z(r, h) - Z(r, l)] < +\infty\)). Consequently, there exist a sequence \(\{r_n\}_{n \in \mathbb{N}}\) such that \(\lim_{n \to +\infty} r_n = +\infty\) (resp. \(\lim_{n \to +\infty} r_n = R\)), and a constant \(C > 0\) such that \(Z(r_n, h) < Z(r_n, l) + C\ \forall n \in \mathbb{N}\). So, by (9), we have

\[
(m - 4)Z(r_n, l) < d \log r_n + O(1).
\]

If we assume \(f \in \mathcal{M}(\mathbb{K})\), then by hypothesis, \(\lim sup |f|(r) = 0\) and so \(l\) is a transcendental function. Thereby, when \(m \geq 5\), we have \(\lim_{n \to +\infty} Z(r_n, l) = +\infty\), a contradiction to (10). Now, if we assume \(f \in \mathcal{M}_u(\overline{\theta} l(0, R^-))\), then at least one of the two functions \(h, l\) belongs to \(\mathcal{A}_u(\overline{\theta} l(0, R^-))\). Since, by hypothesis, \(\lim sup |f|(r) < +\infty\), we deduce that \(l\) must lie in \(\mathcal{A}_u(\overline{\theta} l(0, R^-))\) because if \(l \in \mathcal{A}_u(\overline{\theta} l(0, R^-))\), then \(h \in \mathcal{A}_u(\overline{\theta} l(0, R^-))\) and in this case \(\lim sup |f|(r) = +\infty\), a contradiction. Hence \(\lim_{n \to +\infty} Z(r_n, l) = +\infty\), a contradiction to (10) again.

Thus, when \(m \geq 5\), \(f' + T f^m\) has infinitely many zeros in \(\mathbb{K}\) (resp. in \(\overline{\theta} l(0, R^-)\)) which are not zeros of \(f\). Consequently, by Lemma 1, \(f' + T f^m\) has infinitely many zeros in \(\overline{\theta} l(0, R^-)\) that are not zeros of \(f\).

Now, consider \(T \equiv 1\) and suppose that \(f' + f^4\) has no zeros in \(\mathbb{K}\) which are not zeros of \(f\). Then \(d = -1\). So, by (10), we obtain

\[
0 < - \log r_n + O(1) \quad \forall n \in \mathbb{N},
\]

and hence we have a contradiction when \(n \to +\infty\). Consequently, \(f' + f^4\) has at least one zero in \(\mathbb{K}\) that is not a zero of \(f\).
2.6. Proof of Theorem 4

Proof. Let \( \{r_n\}_{n \in \mathbb{N}} \) be a \( g \)-suitable sequence. For each \( n \in \mathbb{N}^* \), there exists \( r'_n \in ]r_n, r_{n+1}[ \) such that \( g \) has no zero and no pole in the annulus \( \Gamma(0, r_n, r'_n) \). Consequently, by Lemma 2, we have \( \frac{g'(x)}{g(x)} = \frac{1}{x} \) \( \forall x \in \Gamma(0, r_n, r'_n) \). So, \( \nu\left( \frac{g'}{g}, r \right) = -1 \) \( \forall r \in ]r_n, r'_n[ \).

Observe that the poles of \( \frac{g'}{g} \) are all simple ones and correspond to the zeros and the poles of \( g \). Since \( g \) is a transcendental meromorphic function in \( \mathbb{K} \) (resp. \( g \) is unbounded in \( d(0, R^-) \)), we derive that \( \frac{g'}{g} \) has infinitely many poles in \( \mathbb{K} \) (resp. in \( d(0, R^-) \)). Moreover, since \( \nu\left( \frac{g'}{g}, r \right) = -1 \) whenever \( r \in ]r_n, r'_n[ \), by Corollary 5, the difference between the number of poles and the number of zeros of \( \frac{g'}{g} \) in \( d(0, r) \) is just 1. Then clearly, \( \frac{g'}{g} \) has infinitely many zeros in \( \mathbb{K} \) (resp. in \( d(0, R^-) \)).

2.7. Proof of Theorem 5

Proof. Here we assume \( \deg(A) = \deg(B) \). Let \( f \) be of the form \( \frac{h}{l} \) with \( h, l \in A(\mathbb{K}) \) having no common zeros. As in the proofs of Theorem 3, we can write \( h \) in the form \( \frac{\tilde{h}}{u} \), where the zeros of \( \tilde{h} \) are exactly the different zeros of \( h \) but all with multiplicity 1, and \( h' \) is of the form \( hu \) with \( u \in A(\mathbb{K}) \).

Suppose that \( f' + Tf^m \) only has finitely many zeros which are not zeros of \( f \). There exists a \( P \in \mathbb{K}[x] \) such that \( f' + Tf^m = \frac{Pl}{B^{m-1}l} \) with \( P \tilde{h} \) and \( Bl^m \) having no common zeros in \( \mathbb{K} \).

On the other hand, we have

\[
f' + Tf^m = \frac{[Bl^{m-2}(ul - \tilde{h} l') + Ah^{m-1}] \tilde{h}}{Bl^m}.
\]

Since \( h, l \) have no common zeros and since \( A, B \) have no common zeros either, each zero \( \alpha \) of \( [Bl^{m-2}(ul - \tilde{h} l') + Ah^{m-1}] \tilde{h} \) that is not a zero of \( f' + Tf^m \) must be a zero of \( A \) or a zero of \( B \) or \( l \). But note that if \( \alpha \) is a zero of \( l \) then it is a zero of \( A \). Thus the zeros of \( [Bl^{m-2}(ul - \tilde{h} l') + Ah^{m-1}] \tilde{h} \) which are not zeros of \( f' + Tf^m \) must be zeros of \( A \) or \( B \) and therefore are a finite number. Moreover, we notice that a zero \( \alpha \) of \( [Bl^{m-2}(ul - \tilde{h} l') + Ah^{m-1}] \tilde{h} \) is not a zero of \( f \) except if it is a zero of \( B \), because a zero of \( f \) cannot be a zero of \( u \). Consequently, the zeros of \( f' + Tf^m \) that are not zeros of \( f \) are the zeros of \( Bl^{m-2}(ul - \tilde{h} l') + Ah^{m-1} \tilde{h} \) (counting multiplicities), except a finite number.

Next, we may notice that \( h \notin \mathbb{K}[x] \). Indeed, suppose \( h \in \mathbb{K}[x] \). Since \( f \notin \mathbb{K}(x) \), then \( l \notin \mathbb{K}[x] \) and hence \( Bl^{m-2}(ul - \tilde{h} l') + Ah^{m-1} \tilde{h} \notin \mathbb{K}[x] \). Therefore, \( Bl^{m-2}(ul - \tilde{h} l') + Ah^{m-1} \tilde{h} \) has infinitely many zeros which are not zeros of \( f \), a contradiction to our initial supposition.
Now, we consider $H = \frac{f'}{f'} = -\frac{A}{B} + \frac{\tilde{P} h}{B h^m} = -\frac{A}{B} + \frac{P}{B h^{m-1}}$. Since $h$ is not a polynomial in $K$ we have $\lim_{|x| \to +\infty} \frac{P(x)}{B h^{m-1}(x)} = 0$. Moreover, since $\deg(A) = \deg(B)$, $A$ and $B$ have the same number of zeros, taking multiplicities into account and hence we may derive that $\lim_{|x| \to +\infty} \frac{A(x)}{B(x)} = a$ with $a \in \mathbb{R}_+$. Hence, $\lim_{|x| \to +\infty} |H(x)| = a$. Consequently, there exists a $\rho > 0$ such that $\nu(H, r) = 0$ for all $r \geq \rho$.

On the other hand, since $f$ is an optimal function, there exists a $f$-suitable sequence $\{r_n\}_{n \in \mathbb{N}}$ with $\lim_{n \to +\infty} r_n = +\infty$. Let $\{s_n\}_{n \in \mathbb{N}}$ be another sequence such that $r_n < s_n < r_{n+1}$ and such that $\nu(f, r)$ is constant inside $[r_n, s_n]$. By Proposition 20.9 [7], we have $\nu(\frac{r_n}{f'}, r) = \nu(f', r) - \nu(f, r) - 1$ for all $r \in [r_n, s_n]$. Consequently,

$$0 = \nu(H, r) = (1 - m)\nu(f, r) - 1$$

for all $r \in [r_n, s_n]$, a contradiction when $m \geq 3$. Thus, $f' + Tf^m$ has infinitely many zeros in $K$ which are not zeros of $f$.

2.8. Proof of Theorem 6

Proof. As in the proof of Theorem 3, without loss of generality, we may place ourselves in the spherically complete field $\hat{K}$ and consider $f \in M_n(\hat{d}(0, R^-))$. So, there exist functions $h, l \in A(\hat{d}(0, R^-))$ having no common zeros, such that $f = \frac{h}{l}$. Moreover, at least one of them is unbounded. As in the proofs of Theorems 3 and 5, we can write $h$ in the form $\hat{h} h$, where the zeros of $\hat{h}$ are exactly the different zeros of $h$ but all with order 1. Then $h' = \tilde{h} u$ with $u \in A(\hat{d}(0, R^-))$.

Suppose that $f' + U f^m$ only has a finite number of zeros which are not zeros of $f$. The proof now is similar to this of Theorem 5. There exists $P \in \hat{K}[x]$ such that $f' + U f^m$ is of the form $\frac{\tilde{P} h}{\psi l^m}$ with $\tilde{P} h$ and $\psi l^m$ having no common zeros in $\hat{d}(0, R^-)$. Consequently, $f' = \frac{\tilde{P} h}{\psi l^m}$. Since $f$ is an optimal function, there exists a $f$-suitable sequence $\{r_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \to +\infty} r_n = R$. Let $\{s_n\}$ be another sequence such that $r_n < s_n < r_{n+1}$ and such that $\nu(f, r)$ is constant inside $[r_n, s_n]$. By Corollary 5, we have $\nu(f', r) = \nu(f, r) - 1$ for all $r \in [r_n, s_n]$, and by Proposition 20.9 [7], we have $\nu(\frac{r_n}{f'}, r) = \nu(f', r) - \nu(f, r) - 1$ for all $r \in [r_n, s_n]$. Consequently, as in the proof of the previous theorem, we have

$$\nu(\frac{r_n}{f'}, r) = -(m - 1)\nu(f, r) - 1$$

for all $r \in [r_n, s_n]$.\]
On the other hand, considering that \( f \) is of the form \( \frac{\psi h}{l} \) we deduce that

\[
f' + Uf^m = \left[ \psi l^{m-2}(ul - \overline{hl}') + \phi h^m \overline{h}^{m-1} \right] \overline{h}.
\]

Let \( H = \frac{P_{\psi h} - \phi h^m}{\psi l^m} \). We shall prove that \( h \) is unbounded in \( \hat{d}(0, R^-) \). Suppose that \( h \in \mathcal{A}_h(\hat{d}(0, R^-)) \). Since \( \phi \) and \( \psi \) belong to \( \mathcal{A}_h(\hat{d}(0, R^-)) \), then \( H \) belong to \( \mathcal{M}_h(\hat{d}(0, R^-)) \), hence \( l \in \mathcal{A}_l(\hat{d}(0, R^-)) \). Thereby, \( \psi l^{m-2}(ul - \overline{hl}') + \phi h^m \overline{h}^{m-1} \) is an unbounded analytic function in \( \hat{d}(0, R^-) \). So, by Lemma 3, \( \psi l^{m-2}(ul - \overline{hl}') + \phi h^m \overline{h}^{m-1} \) has infinitely many zeros in \( \hat{d}(0, R^-) \), but these zeros are the zeros of \( f' + Uf^m \) that are not zeros of \( f \) except a finite number of them (see arguments in the proof of Theorem 5), a contradiction to our supposition. Hence, we may deduce that

\[
\lim_{r \to R^-} \left| \frac{P}{\psi h^{m-1}} \right|(r) = 0.
\]

Now, since \( \phi \) and \( \psi \) have the same finite number of zeros in \( \hat{d}(0, R^-) \) (counting multiplicities), there exists a \( \rho < R \) such that \( \nu(\frac{\phi}{\psi}, r) = 0 \) \( \forall r \in [\rho, R] \), and therefore \( |U|(r) \) is a constant \( c \) in \( [\rho, R] \). Consequently, there exists a \( \rho' \in [\rho, R] \) such that \( |H|(r) = |U|(r) = c \) \( \forall r \in [\rho', R] \). Thus, \( \nu(H, r) = 0 \) \( \forall r \in [\rho', R] \).

The end of the proof is then similar to that of the previous theorem. By (11) and the previous observation, we have \( (m-1)\nu(f, r) = -1 \) \( \forall r \in ]r_n, s_n[ \) \( \forall n \in \mathbb{N} \), which is absurd because \( m \geq 3 \). Hence \( f' + Uf^m \) has infinitely many zeros which are not zeros of \( f \) whenever \( m \geq 3 \).

\[ \blacksquare \]

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**References**


Jacqueline Ojeda  
Laboratoire de Mathématiques (UMR 6620)  
Université Blaise Pascal  
Campus Universitaire des Cèzeaux  
63177 Aubiere Cedex  
France  
E-mail: Jacqueline.Ojeda@math.univ-bpclermont.fr