ON QUASI-ARMENDARIZ MODULES

Muhittin Başer and M. Tamer Koşan

Abstract. In this paper, we introduce the concept of a \((\alpha\)-quasi-Armendariz \) module, principally quasi-Baer module and study its some properties. In particular, we show: (1) For an \(\alpha\)-quasi-Armendariz module \(M_R\), \(M_R\) is a principally quasi-Baer module if and only if \(M[x; \alpha]_R[x; \alpha]\) is a principally quasi-Baer module. (2) A necessary and sufficient condition for a trivial extensions to be quasi-Armendariz is obtained. Consequently, new families of quasi-Armendariz rings are presented.

1. INTRODUCTION

Throughout this work all rings \(R\) are associative with identity and modules are unital right \(R\)-modules and \(\alpha : R \rightarrow R\) is an endomorphism of the ring \(R\). In [7] Clark called a ring \(R\) quasi-Baer ring if the right annihilator of each right ideal of \(R\) is generated (as a right ideal) by an idempotent. Recently, Birkenmeier et al. [4] called a ring \(R\) right (resp. left) principally quasi-Baer [or simply right (resp. left) p.q.-Baer] if the right (resp. left) annihilator of a principal right (resp. left) ideal of \(R\) is generated by an idempotent. \(R\) is called p.q.-Baer if it is both right and left p.q.-Baer. A ring \(R\) is called a right (resp. left) p.p.-ring if the right (resp. left) annihilator of every element of \(R\) is generated by an idempotent. \(R\) is called a p.p.-ring if it is both a right and left p.p.-ring. A ring is called reduced ring if it has no nonzero nilpotent elements and \(M_R\) is called \(\alpha\)-reduced module by Lee-Zhou [13] if, for any \(m \in M\) and \(a \in R\), (1) \(ma = 0\) implies \(mR \cap Ma = 0\), (2) \(ma = 0\) iff \(m\alpha(a) = 0\), where \(\alpha : R \rightarrow R\) is a ring endomorphism with \(\alpha(1) = 1\). The module \(M_R\) is called a reduced module if \(M\) is \(1_R\)-reduced. It is clear that \(R\) is a reduced ring if \(R_R\) is a reduced module.

In [13] Lee-Zhou introduced the following notation. For a module \(M_R\), we consider \(M[x; \alpha] = \left\{ \sum_{i=0}^{s} m_i x^i : s \geq 0, m_i \in M \right\}\). This set is an abelian group

Received January 28, 2006, accepted September 14, 2006.
Communicated by Wen-Fong Ke.
2000 Mathematics Subject Classification: 16D80.
Key words and phrases: (Quasi)-Armendariz module, (Quasi)-Baer module, p.p.-module.
under an obvious addition operation. Moreover \( M[x; \alpha] \) becomes a module over \( R[x; \alpha] \) under the following scalar product operation:

For \( m(x) = \sum_{i=0}^{s} m_i x^i \in M[x; \alpha] \) and \( f(x) = \sum_{j=0}^{t} a_j x^j \in R[x; \alpha] \), \( m(x)f(x) = \sum_{k=0}^{s+t} \left( \sum_{i+j=k} m_i \alpha^i(a_j) \right) x^k \).

The modules \( M[x; \alpha] \) is called the \textit{skew polynomial extension} of \( M \). When \( \alpha \) is identity, we write \( M[x]_{R[\alpha]} \) for \( M[x; 1_R]_{R[\alpha]} \).

According to Lee-Zhou [13] a module \( M_R \) is called \( \alpha \)-\textit{Armendariz} if the following conditions are satisfied:

1. For \( m \in M \) and \( a \in R \), \( ma = 0 \) if and only if \( m\alpha(a) = 0 \),
2. For any \( m(x) = \sum_{i=0}^{s} m_i x^i \in M[x; \alpha] \) and \( f(x) = \sum_{j=0}^{t} a_j x^j \in R[x; \alpha] \), \( m(x)f(x) = 0 \) implies \( m_i \alpha^i(a_j) = 0 \) for all \( i \) and \( j \).

The module \( M_R \) is \textit{Armendariz} iff \( M_R \) is 1-R-Armendariz. If \( M_R \) is \( \alpha \)-reduced then \( M_R \) is \( \alpha \)-Armendariz.

For a subset \( X \) of a module \( M_R \), let \( r_R(X) = \{ r \in R : Xr = 0 \} \). In [13] Lee-Zhou introduced Baer modules, quasi-Baer modules and \( p.p. \)-modules as follows.

1. \( M_R \) is called \textit{Baer} if, for any subset \( X \) of \( M \), \( r_R(X) = eR \) where \( e^2 = e \in R \).
2. \( M_R \) is called \textit{quasi-Baer} if, for any submodule \( N \) of \( M \), \( r_R(N) = eR \) where \( e^2 = e \in R \).
3. \( M_R \) is called \textit{principally projective} (or simply \( p.p. \)) if, for any \( m \in M \), \( r_R(m) = eR \) where \( e^2 = e \in R \).

2. \textbf{QUASI-ARMENDARIZ MODULES AND PRINCIPALLY QUASI-BAER MODULES}

Our focus in this section is to introduce the concept of a \((\alpha-)\) quasi-Armendariz module, principally quasi-Baer module and study its some properties. It is easy to see that the notation of quasi-Armendariz modules generalize that of Armendariz modules as well as that \( \alpha \)-reduced modules. We investigate connections to other related conditions.

Following [16] a ring \( R \) is called \textit{Armendariz} if, for any polynomials \( f(x) = \sum_{i=0}^{m} a_i x^i \) and \( g(x) = \sum_{j=0}^{n} b_j x^j \in R[x] \), \( f(x)g(x) = 0 \) implies \( a_i b_j = 0 \) for all \( i \) and \( j \). This notion is generalized by Hirano [8] as the follows; a ring \( R \) is called \textit{quasi-Armendariz} if, whenever \( f(x)R[x]g(x) = 0 \), where \( f(x) = \sum_{i=0}^{m} a_i x^i \), \( g(x) = \sum_{j=0}^{n} b_j x^j \in R[x] \) then \( a_i R b_j = 0 \) for all \( i \) and \( j \).

Armendariz rings are quasi-Armendariz. A commutative ring \( R \) is Armendariz if and only if it is quasi-Armendariz. The following example shows that there exists a quasi-Armendariz ring \( R \) such that \( R \) is not Armendariz.
Example 2.1. Let $F$ be a field and consider the ring

$$R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}.$$  

Then by ([11], Example 1), $R$ is not Armendariz. Since $F$ is a quasi-Armendariz, $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ is a quasi-Armendariz by [8, Corollary 3.15].

Following Anderson and Camillo [1], a right $R$ module $M$ is called an Armendariz module if, whenever $m(x)f(x) = 0$ where $m(x) = \sum_{i=0}^{n} m_{i}x^{i} \in M[x]$ and $f(x) = \sum_{j=0}^{s} a_{j}x^{j} \in R[x]$, then $m_{i}a_{j} = 0$ for all $i$ and $j$. Similarly one can define an Armendariz left $R$-module. Generalizing this definition, we begin the following.

**Definition 2.2.** A right $R$-module $M$ is called quasi-Armendariz if, whenever $m(x)R[x]f(x) = 0$ where $m(x) = \sum_{i=0}^{n} m_{i}x^{i} \in M[x]$ and $f(x) = \sum_{j=0}^{s} a_{j}x^{j} \in R[x]$, then $m_{i}Ra_{j} = 0$ for all $i$ and $j$.

Clearly, $R$ is a quasi-Armendariz ring if and only if $R_{R}$ is a quasi-Armendariz right $R$-module and Armendariz modules are quasi-Armendariz.

**Example 2.3.** Several easy examples of quasi-Armendariz modules can be given:

1. Every reduced module is a quasi-Armendariz module. (2) For any $n \in \mathbb{Z}$, $\mathbb{Z}_{n}$ is a quasi-Armendariz $\mathbb{Z}$-module.

**Lemma 2.4.** Let $M$ be an $R$-module.

1. The following are equivalent:

   (a) For any $m(x) \in M[x]$, $(r_{R[x]}(m(x)R[x]) \cap R)[x] = r_{R[x]}(m(x)R[x]).$

   (b) For any $m(x) = \sum_{i=0}^{n} m_{i}x^{i} \in M[x]$ and $f(x) = \sum_{j=0}^{s} a_{j}x^{j} \in R[x]$, $m(x)R[x]f(x) = 0$ implies $m_{i}Ra_{j} = 0$.

2. Let $M_{R}$ be a quasi-Armendariz module and $m(x) \in M[x]$. If $r_{R[x]}(m(x)R[x]) \neq 0$, then $r_{R[x]}(m(x)R[x]) \cap R \neq 0$.

**Proof.** (1) (a) $\Rightarrow$ (b) Let $m(x) = \sum_{i=0}^{n} m_{i}x^{i} \in M[x]$ and $f(x) = \sum_{j=0}^{s} a_{j}x^{j} \in R[x]$ be such that $m(x)R[x]f(x) = 0$. Then $f(x) \in r_{R[x]}(m(x)R[x])$. By (a) $f(x) \in (r_{R[x]}(m(x)R[x]) \cap R)[x]$, and so $a_{j} \in r_{R[x]}(m(x)R[x]) \cap R$ for all $j = 0, 1, \ldots, s$. Therefore, $m(x)R[x]a_{j} = 0$ and so $m_{i}Ra_{j} = 0$ for all $i$ and $j$.

(b) $\Rightarrow$ (a) Let $g(x) = \sum_{j=0}^{s} b_{j}x^{j} \in (r_{R[x]}(m(x)R[x]) \cap R)[x]$. Then $b_{j} \in r_{R[x]}(m(x)R[x])$ and so $m(x)R[x]b_{j} = 0$ for all $j$. Then $m(x)R[x]g(x) = 0$. Hence $g(x) \in r_{R[x]}(m(x)R[x])$. Therefore $(r_{R[x]}(m(x)R[x])) \cap R[x] \subseteq r_{R[x]}(m(x))$. 

On Quasi-Armendariz Modules
Let $h(x) = \sum_{j=0}^{k} c_j x^j \in r_{R[x]}(m(x)R[x])$. Then $m(x)R[x]h(x) = 0$. By (b) $m_i R e_j = 0$. Therefore $m(x)R[x]c_j = 0$ for all $j$. Hence $c_j \in r_{R[x]}(m(x)R[x]) \cap R$ for all $j$, and so $h(x) \in (r_{R[x]}(m(x)R[x]) \cap R)[x]$. Thus $r_{R[x]}(m(x)R[x]) \subseteq (r_{R[x]}(m(x)R[x]) \cap R)[x]$. Hence $(r_{R[x]}(m(x)R[x]) \cap R)[x] = r_{R[x]}(m(x)R[x])$.

(2) Clear from (1) $(b) \Rightarrow (a)$. ■

A generalization of a zero commutative ring is a semicommutative ring. A ring $R$ is semicommutative if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Historically, some of the earliest results known to us about semicommutative rings (although not so called at the time) was due to Shin [17].

McCoy [15] proved that if $R$ is a commutative ring, then whenever $g(x)$ is a zero-divisor in $R[x]$, there exists a non-zero element $c \in R$ such that $cg(x) = 0$ and Hirano [8] proved that if $R$ is a semi-commutative ring, then whenever $f(x)$ is a zero-divisor in $R[x]$ there exists a non-zero element $c \in R$ such that $f(x)c = 0$.

We shall extend these results to module case.

**Proposition 2.5.** Let $M$ be a reduced module. If $m'(x)$ is a torsion element in $M[x]$ (i.e. $m'(x)h(x) = 0$ for some $0 \neq h(x) \in R[x]$), then there exists a non-zero element $c$ of $R$ such that $m'(x)c = 0$.

**Proof.** Let $m'(x) = \sum_{i=0}^{n} m_i x^i$ and $h(x) = \sum_{j=0}^{s} h_j x^j$ and $m'(x)h(x) = 0$.

Then

(1) $m_0 h_0 = 0$ ;
(2) $m_0 h_1 + m_1 h_0 = 0$ ;
(3) $m_0 h_2 + m_1 h_1 + m_2 h_0 = 0$ ;
\vdots
\vdots
(n + s) $m_n h_s = 0$.

Note that for a reduced module $M$ for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mRa = 0$ and $ma^2 = 0$ implies $ma = 0$ by Lemma 1.2 in [13]. By (1) $m_0 R h_0 = 0$ since $M$ is reduced. Multiplying (2) by $h_0$ from the right and using hypothesis we obtain $m_1 Rh_0 = 0$ and so $m_0 Rh_1 = 0$. Multiplying (3) by $h_0$ from the right and using hypothesis, from (1) and (2), we have $m_2 h_0 = 0, m_1 h_1 = 0, m_0 h_2 = 0$, and so $m_2 Rh_0 = 0, m_1 Rh_1 = 0, m_0 Rh_2 = 0$. By induction, $m_i Rh_j = 0$ for all $i$ and $j$. Assume that $h(x) \neq 0$. Then at least one of coefficients of $h(x)$ is nonzero, say $h_{j_0} \neq 0$. Then $m'(x)h_{j_0} = 0$. This completes the proof. ■

Now, we give the following new definition which is connected with Lee-Zhou definitions.
Definition 2.6. The module $M$ is called principally quasi-Baer module (p.q.-Baer for short) if, for any $m \in M$, $r_R(mR) = eR$ where $e^2 = e \in R$.

It is clear that $R$ is a right p.q.-Baer ring iff $R_R$ is a p.q.-Baer module. If $R$ is a p.q.-Baer ring, then for any right ideal $I$ of $R$, $I_R$ is a p.q.-Baer module. Every submodule of a p.q.-Baer module is p.q.-Baer module. Moreover, every quasi-Baer module is p.q.-Baer, and every Baer module is quasi-Baer. If $R$ is commutative then $M_R$ is a p.q.-module iff $M_R$ is a p.q.-Baer module.

We can give the following definition by considering definition of $\alpha$-Armendariz module.

$M_R$ is called $\alpha$-quasi-Armendariz if the following conditions are satisfied:

1. For any $m \in M$ and any $a \in R$, $ma = 0$ if and only if $ma(a) = 0$,
2. For any $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x; \alpha]$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in R[x; \alpha],
   m(x)R[x; \alpha]f(x) = 0$ implies $m_i R \alpha^t(a_j) = 0$ for all $i$ and $j$.

Note that the module $M_R$ is quasi-Armendariz if and only if $M_R$ is a $1_R$-quasi-Armendariz.

Theorem 2.7. Let $M$ be an $\alpha$-quasi-Armendariz module. Then $M_R$ is a p.q.-Baer module if and only if $M[x; \alpha]_R[x; \alpha]$ is a $p.q.-Baer module$.

Proof. Assume that $M[x; \alpha]_R[x; \alpha]$ is a $p.q.-Baer module$. Let $m \in M$. Then there exists an idempotent $f(x) \in R[x; \alpha]$ such that $r_{R[x; \alpha]}(mR[x; \alpha]) = f(x)R[x; \alpha]$. Note that $f(x)R[x; \alpha] \subseteq r_{R[x; \alpha]}(mR) = r_{R}(mR[x; \alpha])$ always holds. Let $g(x) = b_0 + \ldots + b_t x^t \in r_{R}(mR[x; \alpha])$. Then $mR \beta_j = 0$ for all $0 \leq j \leq t$. By hypothesis $mR \alpha^j(b_j) = 0$ for all $i$ and $0 \leq j \leq t$. Let $h(x) = \sum_{k=0}^{s} c_k x^k \in R[x; \alpha]$. Then $mh(x) b_j = \sum_{k=0}^{s} m_k c_k \alpha^j(b_j) x^k = 0$ for all $j$, and so $m(x) g(x) = 0$ for all $h(x) = \sum_{k=0}^{s} c_k x^k \in R[x; \alpha]$. Hence $g(x) \in r_{R[x; \alpha]}(mR[x; \alpha])$. Thus $r_{R[x; \alpha]}(mR[x; \alpha]) = f(x)R[x; \alpha] = r_{R}(mR)[x; \alpha]$. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ where all $a_i \in r_{R}(mR)$. Note that, for any $a \in r_{R}(mR)$, $f(x)a = a$. Hence $f(x)a = (a_0 + a_1 x + \ldots + a_n x^n)a = a_0 a + a_1 xa + \ldots + an x^n a = a$ implies that $a = a_0 a$. Since $a_0^2 = a_0$ and $r_{R}(mR) = a_0 R$, $M_R$ is a $p.q.-Baer module$.

For the converse, assume that $M_R$ is a $p.q.-Baer$. Let $m(x) = m_0 + m_1 x + \ldots + m_n x^n \in M[x; \alpha]$. Then $r_{R[x]}(m(x)R[x]) = r_{R[x]}(m(x)R[x]) \cap R[x] = r_R(m(x)R[x])$ by Lemma 2.4. Let $C_{mR}$ the set of all coefficients of $m(x)R[x]$, i.e., $C_{mR} = \{m_i R : i = 0, \ldots, n\}$. $r_{R[x]}(m(x)R[x]) \cap R = r_{R}(m(x)R[x]) = r_R(C_{mR})$. Since $M_R$ is a $p.q.-Baer$, $r_{R}[C_{mR}] = \cap_{i=0}^{n} r_{R}(m_i R) = \cap_{i=0}^{n} e_i R$, where $e_i^2 = e_i \in R$ and $r_{R}(m_i R) = e_i R$. We claim that $\cap_{i=0}^{n} e_i R = e R$, where $e^2 = e \in R$. Since $m_1 R e_1 = 0$, $m_1 R e_0 e_1 = 0$ and so $e_0 e_1 \in r_{R}(m_1 R) = e_1 R$. 

On Quasi-Armendariz Modules 577
Thus \( e_1e_0e_1 = e_0e_1 \). Let \( f_1 = e_0e_1 \) then \( f_1^2 = (e_0e_1)(e_0e_1) = e_0e_1 = f_1 \) and \( e_0R \cap e_1R = f_1R \). Since \( m_2R e_2 = 0 \) and so \( f_1e_2 = e_2R \). Hence \( e_2f_1e_2 = f_1e_2 \). Let \( f_2 = f_1e_2 \). Then \( f_2^2 = f_2 \) and \( f_1R \cap e_2R = f_2R \).

Continuing this process, we obtain \( f_n^2 = f_n \) and so \( \cap_{i=0}^n f_iR = f_nR \). Thus \( r_{R[x;\alpha]}(m(x)R[x;\alpha]) = r_R(C_mR[x;\alpha]) = f_nR[x;\alpha] \).

**Theorem 2.8.** Let \( M_R \) be a reduced module. Then the following statements are equivalent;

1. \( M_R \) is a p.p.-module.
2. \( M_R \) is a p.q.-Baer module.
3. \( M[x]_{R[x]} \) is a p.p.-module.
4. \( M[x]_{R[x]} \) is a p.q.-Baer module.

**Proof.** (1) \( \Leftrightarrow \) (3) By [13, Corollary 2.12].
(2) \( \Leftrightarrow \) (4) Clear by Theorem 2.7 since every reduced module is quasi-Armendariz.
(1) \( \Leftrightarrow \) (2) Let \( m \in M \). If \( a \in r_R(m) \) then \( ma = 0 \) and by [13, Lemma 1.2], \( mRa = 0 \) and so \( a \in r_R(mR) \). Then \( r_R(m) \subseteq r_R(mR) \). But \( r_R(mR) \subseteq r_R(m) \) obviously holds. Consequently, \( r_R(mR) = r_R(m) = eR \). Hence the claim follows.

3. WHEN IS A TRIVIAL EXTENSION QUASI-ARMENDARIZ?

Given a ring \( R \) and a bimodule \( R M_R \), the trivial extension of \( R \) by \( M \) is the ring \( T(R, M) = R \oplus M \) with the usual addition and multiplication

\[
(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).
\]

This is isomorphic to the ring of all matrices \( \begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \) where \( a \in R, m \in M \).

**Lemma 3.1.** ([14, Lemma 2.1]) Let \( M \) be an \( (R, R) \)-bimodule. Then \( M[x] \) is an \( (R[x], R[x]) \)-bimodule and \( T(R[x], M[x]) = T(R, M)[x] \).

**Proposition 3.2.** Let \( M \) be an \( (R, R) \)-bimodule. If the trivial extension \( T(R, M) \) is a quasi-Armendariz ring, then \( M \) is a quasi-Armendariz left and right \( R \)-module.

**Proof.** Let \( m(x) = m_0 + m_1x + \ldots + m_x x^n \in M[x] \), \( f(x) = a_0 + a_1x + \ldots + a_n x^n \in R[x] \) and suppose that \( f(x) R[x] m(x) = 0 \). For an arbitrary \( c \in R, n \in M \)
we have the following equation:

\[
\left( \sum_{i=0}^{n} \left( \begin{array}{cc} a_i & 0 \\ 0 & a_i \end{array} \right) x^i \right) \left( \begin{array}{cc} c & n \\ 0 & c \end{array} \right) \left( \sum_{j=0}^{s} \left( \begin{array}{cc} 0 & m_j \\ 0 & 0 \end{array} \right) x^j \right) \\
= \left( \begin{array}{cc} f(x) & 0 \\ 0 & f(x) \end{array} \right) \left( \begin{array}{cc} c & n \\ 0 & c \end{array} \right) \left( \begin{array}{cc} 0 & m(x) \\ 0 & 0 \end{array} \right) \\
= \left( \begin{array}{cc} f(x)c & f(x)n \\ 0 & m(x) \end{array} \right) \left( \begin{array}{cc} 0 & m(x) \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & f(x)cm(x) \\ 0 & 0 \end{array} \right) = 0.
\]

Since \( T(R, M) \) is quasi-Armendariz,

\[
\left( \begin{array}{cc} a_i & 0 \\ 0 & a_i \end{array} \right) \left( \begin{array}{cc} c & n \\ 0 & c \end{array} \right) \left( \begin{array}{cc} 0 & m_j \\ 0 & 0 \end{array} \right) = 0
\]

for all \( i \) and \( j \). Therefore \( a_i cm_j = 0 \) for all \( i \) and \( j \). Consequently, \( M \) is a quasi-Armendariz left \( R \)-module. Similarly, \( M \) is a quasi-Armendariz right \( R \)-module.

Letting \( RMR = R R \) yields the following:

**Corollary 3.3.** If the trivial extension \( T(R, R) \) is a quasi-Armendariz ring, then also \( R \) is quasi-Armendariz.

**Theorem 3.4.** Let \( M \) be an \((R, R)\)-bimodule such that

1. \( R \) is a quasi-Armendariz ring.
2. \( M \) is an Armendariz left and quasi-Armendariz right \( R \)-module.
3. If \( f(x)Rg(x) = 0 \) in \( R[x] \), then \( f(x)M[x] \cap M[x]g(x) = 0 \).

Then the trivial extension \( T(R, M) \) is a quasi-Armendariz ring.

**Proof.** Suppose that \( \alpha(x)T(R, M)\beta(x) = 0 \) where

\[
\alpha(x) = \left( \begin{array}{cc} a_0 & m_0 \\ 0 & a_0 \end{array} \right) + \left( \begin{array}{cc} a_1 & m_1 \\ 0 & a_1 \end{array} \right) x + \ldots + \left( \begin{array}{cc} a_n & m_n \\ 0 & a_n \end{array} \right) x^n \in T(R, M)[x],
\]

\[
\beta(x) = \left( \begin{array}{cc} b_0 & l_0 \\ 0 & b_0 \end{array} \right) + \left( \begin{array}{cc} b_1 & l_1 \\ 0 & b_1 \end{array} \right) x + \ldots + \left( \begin{array}{cc} b_s & l_s \\ 0 & b_s \end{array} \right) x^s \in T(R, M)[x],
\]

Let \( f(x) = a_0 + a_1 x + \ldots + a_n x^n \), \( g(x) = b_0 + b_1 x + \ldots + b_s x^s \),

\( m(x) = m_0 + m_1 x + \ldots + m_n x^n \), \( l(x) = l_0 + l_1 x + \ldots + l_s x^s \).
Then \( f(x), g(x) \in R[x] \) and \( m(x), l(x) \in M[x] \). For an arbitrary \( \begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \in T(R, M) \), it follows that

\[
0 = \begin{pmatrix} f(x) & m(x) \\ 0 & f(x) \end{pmatrix} \begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \begin{pmatrix} g(x) & l(x) \\ 0 & g(x) \end{pmatrix} \\
= \begin{pmatrix} f(x)ag(x) & f(x)al(x) + f(x)mg(x) + m(x)ag(x) \\ 0 & f(x)ag(x) \end{pmatrix}.
\]

Thus \( f(x)ag(x) = 0 \) and \( f(x)al(x) + f(x)mg(x) + m(x)ag(x) = 0 \). Since \( a \in R \) arbitrary, \( f(x)Rg(x) = 0 \). Since \( R \) is a quasi-Armendariz by (1), \( a_i R b_j = 0 \) for all \( i \) and \( j \). Since \( f(x)[al(x) + mg(x)] + [m(x)a]g(x) = 0 \), \( f(x)[al(x) + mg(x)] = -[m(x)a]g(x) \in f(x)M[x] \cap M[x]g(x) = 0 \), so \( f(x)[al(x) + mg(x)] = [m(x)a]g(x) = 0 \). Since \( a \in R \) arbitrary \( m(x)Rg(x) = 0 \). Then by (2), \( m_i R b_j = 0 \) for all \( i \) and \( j \). And \( f(x)[al(x)] = -[f(x)m_i]g(x) \in f(x)M[x] \cap M[x]g(x) = 0 \) by (3). So \( f(x)al(x) = 0 \) and hence \( f(x)Rl(x) = 0 \). Then by (2), \( M \) is an Armendariz left \( R \)-module and hence \( M \) is a quasi-Armendariz left \( R \)-module. Therefore \( a_i R l_j = 0 \) for all \( i \) and \( j \). For arbitrary \( m \in M \), we have \( f(x)mg(x) = 0 \). But if \( f(x)m \in M[x] \) and since \( M \) is an Armendariz left \( R \)-module by (2), we obtain \( a_i mb_j = 0 \) for all \( i \) and \( j \). Therefore

\[
\begin{pmatrix} a_i & m_i \\ 0 & a_i \end{pmatrix} \begin{pmatrix} c & n \\ 0 & c \end{pmatrix} \begin{pmatrix} b_j & l_j \\ 0 & b_j \end{pmatrix} = \begin{pmatrix} a_i cb_j & a_i c l_j + a_i n b_j + m_i c b_j \\ 0 & a_i cb_j \end{pmatrix} = 0
\]

for all \( i, j \) and \( \begin{pmatrix} c & n \\ 0 & c \end{pmatrix} \in T(R, M) \). Consequently the trivial extension \( T(R, M) \) is a quasi-Armendariz ring.

**Acknowledgment**

We would like to thanks the referee for valuable suggestions which improved the paper considerable.

**References**


M. Tamer Koşan
Department of Mathematics,
Faculty of Science,
Gebze Institute of Technology,
Çaylırova Campus,
41400 Gebze Kocaeli,
Turkey
E-mail: mtkosan@aku.edu.tr