STRONG CONVERGENCE THEOREMS OF REICH TYPE ITERATIVE SEQUENCE FOR NON-SELF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

S. S. Chang, Yeol Je Cho and Y. X. Tian

Abstract. The purpose of this paper is to give some necessary and sufficient conditions for the iterative sequence of Reich type to converging to a fixed point. The results presented in this paper extend and improve some recent results ([2-4]).

1. Introduction

Throughout this paper, we assume that $E$ is a real Banach space, $C$ is a nonempty closed convex subset of $E$, $E^*$ is the dual space of $E$ and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{ f \in E^*, \langle x, f \rangle = ||x|| \cdot ||f||, ||x|| = ||f|| \}, \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

In the sequel, we shall denote the single-valued normalized duality mapping $J$ by $j$ and denote the fixed point set of a mapping $T$ by $F(T)$. If $\{x_n\}$ is a sequence in $E$, then $x_n \rightharpoonup x$ (resp., $x_n \to x$) will denote strong (resp., weak) convergence of the sequence $\{x_n\}$ to a point $x \in E$.

Recall that a mapping $T : C \to C$ is said to be asymptotically nonexpansive ([1]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$||T^nx - T^ny|| \leq k_n ||x - y||, \forall x, y \in C, n \geq 0.$$
A mapping $T$ is said to be uniformly $L$-Lipschitzian if there exists a constant $L > 0$ such that
\[ ||T^n x - T^n y|| \leq L||x - y||, \quad \forall x, y \in C, \quad n \geq 0.\]

It is clear that every non-expansive mapping is asymptotically non-expansive and every asymptotically non-expansive is uniformly $L$-Lipschitzian with a constant $L = \sup_{n \geq 0} k_n \geq 1$. The converses do not hold. The asymptotically non-expansive mappings are important generalization of non-expansive mappings.

**Definition 1.1.** Let $E$ be a real Banach space and $C$ be a nonempty subset of $E$.

1. A mapping $P$ from $E$ onto $C$ is said to be a **retraction** if $P^2 = P$.
2. If there exists a continuous retraction $P : E \to C$ such that $Px = x$ for all $x \in C$, then the set $C$ is said to be a **retract** of $E$.
3. Especially, if there exists a non-expansive retraction $P : E \to C$ such that $Px = x$ for all $x \in C$, then the set $C$ is said to be a **non-expansive retract** of $E$.

**Definition 1.2.** Let $E$ be a real Banach space, $C$ be a nonempty non-expansive retract of $E$ with a non-expansive retraction $P : E \to C$. Let $T : C \to E$ be a non-self mappings.

1. $T$ is said to be **non-self asymptotically nonexpansive** if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that
\[ ||T(PT)^{n-1} x - T(PT)^{n-1} y|| \leq k_n||x - y||, \quad \forall x, y \in C, \quad n \geq 1.\]
2. $T$ is said to be **non-self uniformly $L$-Lipschitzian** if there exists a constant $L > 0$ such that
\[ ||T(PT)^{n-1} x - T(PT)^{n-1} y|| \leq L||x - y||, \quad \forall x, y \in C, \quad n \geq 1.\]

**Proposition 1.1.** If $T : C \to E$ is a non-self asymptotically non-expansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ ($n \to \infty$), then we have the following:

1. The mapping $PT : C \to C$ is an asymptotically non-expansive mapping with the same sequence $\{k_n\}$.
2. $PT$ is a uniformly $L$-Lipschitzian mapping with $L = \sup_{n \geq 1} k_n \geq 1$. 

Proof. In fact, since \( T : C \to E \) is a non-self asymptotically non-expansive mapping, for any \( n \geq 1 \) and \( x, y \in C \), we have
\[
||(PT)^n x - (PT)^n y|| \leq ||T(PT)^{n-1} x - T(PT)^{n-1} y|| \leq k_n ||x - y||.
\]
Thus the conclusion (1) is proved. The conclusion (2) is obvious. This completes the proof.

Definition 1.3. Let \( S := \{ x \in E : ||x|| = 1 \} \) be the unit sphere of a Banach space \( E \). The space \( E \) is said to have a uniformly Gâteaux differentiable norm if, for each \( y \in S \), the limit
\[
\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t}
\]
is attained uniformly for all \( x \in S \).

Remark 1.1. It is well-known that, if \( E \) has a uniformly Gâteaux differentiable norm, then the normalized duality mapping \( J : E \to 2^{E^*} \) is uniformly continuous from the norm topology of \( E \) to the weak* topology of \( E^* \) on any bounded subsets of \( E \).

Let \( K \) be a nonempty closed convex and bounded subset of \( E \) and the diameter of \( K \) be defined by \( d(K) = \sup \{ ||x - y|| : x, y \in K \} \). For each \( x \in K \), let \( r(x, K) = \sup \{ ||x - y|| : y \in K \} \) and \( r(K) = \inf \{ r(x, K) : x \in K \} \) denote the Chebyshev radius of \( K \) relative to itself. The normal structure coefficient \( N(E) \) of \( E \) is defined by
\[
N(E) : = \inf \{ \frac{d(K)}{r(K)} : K \text{ is a closed convex bounded subset of } E \text{ with } d(K) > 0 \}.
\]

A Banach space \( E \) such that \( N(E) > 1 \) is said to have uniformly normal structure. It is known that every space with a uniformly normal structure is reflexive and all uniformly convex and uniformly smooth Banach spaces have uniformly normal structure (see [12]).

Recall that a linear continuous functional \( \mu \in (l^\infty)^* \) is called a Banach limit if
\[
||\mu|| = 1, \quad \lim inf_{n \to \infty} a_n \leq \mu_n(a_n) \leq \lim sup_{n \to \infty} a_n, \quad \mu_n(a_n) = \mu_n(a_{n+1})
\]
for all \( a = \{a_n\} \in l^\infty \), where \( \mu(a), a = \{a_n\} \in l^\infty \) is denoted by \( \mu_n(a_n) \).

Next, we introduce the concept of Reich type iterative sequence for non-self asymptotically nonexpansive mappings.

Definition 1.4. Let \( E \) be a real Banach space and \( C \) be a nonempty closed convex subset of \( E \). Let \( u \in C \) be a given point and \( T : C \to E \) be a non-self
asymptotically nonexpansive mapping with a sequence \( \{k_n\} \subset [1, \infty) \) with \( k_n \to 1 \), then the sequence \( \{x_n\} \) in \( E \) defined by

\[
\begin{cases}
  x_0 \in C, \\
  x_{n+1} = \alpha_n u + (1 - \alpha_n) \frac{1}{n + 1} \sum_{j=0}^{n} (PT)^j y_n, \quad \forall \ n \geq 0, \\
  y_n = \beta_n x_n + (1 - \beta_n) \frac{1}{n + 1} \sum_{j=0}^{n} (PT)^j x_n,
\end{cases}
\]

is called the first type iterative sequence of Reich, where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are two sequences in \([0, 1]\).

In (1.2), taking \( \beta_n = 1 \) for all \( n \geq 0 \), then the sequence \( \{x_n\} \) in \( E \) defined by

\[
\begin{cases}
  x_0 \in C, \\
  x_{n+1} = \alpha_n u + (1 - \alpha_n) \frac{1}{n + 1} \sum_{j=0}^{n} (PT)^j x_n, \quad \forall \ n \geq 0,
\end{cases}
\]

is called the second type iterative sequence of Reich, where \( \{\alpha_n\} \) is a sequence in \([0, 1]\).

Next, we consider some special cases of (1.2) and (1.3).

In 1980 and 1983, Reich [4], [5] proved that if \( E \) is a uniformly smooth Banach space, \( C \) is a weakly compact convex subset of \( E \) with fixed point property for nonexpansive mappings, \( T : C \to C \) is a nonexpansive mapping and \( \alpha_n = n^{-a}, a \in (0, 1) \), then the sequence \( \{x_n\} \) defined by

\[
\begin{cases}
  x_0 \in C, \\
  x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall \ n \geq 0,
\end{cases}
\]

converges strongly to a fixed point of \( T \).

In 1992, Wittmann [6] proved that, if \( E \) is a Hilbert space and the sequence \( \{\alpha_n\} \subset [0, 1] \) satisfies the following conditions:

\[
\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,
\]

then the sequence (1.4) converges strongly to some fixed point of \( T \).

Recently, Zhang [8], [9] and Zeng [10] studied the convergence problem of the following iterative sequence for asymptotically non-expansive mapping \( T : C \to C \):

\[
\begin{align*}
    x_{n+1} &= \alpha_n u + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j y_n, \\
y_n &= \beta_n x_n + (1 - \beta_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n,
\end{align*}
\]

which extended the corresponding results of Reich [4], [5], Shioji and Takahashi [7], Wittmann [6].

In order to prove our main results, we need the following Lemmas.

**Lemma 1.2.** Let \( E \) be a Banach space with uniformly normal structure, \( C \) be a nonempty bounded subset of \( E \) and \( T : C \to C \) be a uniformly \( L \)-Lipschitzian mapping with \( L < \sqrt{N(E)} \). Suppose further that there exists a nonempty bounded closed convex subset \( K \) of \( C \) with the following property (A):

(A) \( x \in K \) implies \( \omega_w(x) \subset K \)

where \( \omega_w(x) \) is the weak \( \omega \)-limit set of \( T \) at \( x \), i.e.,

\[
\omega_w(x) = \{ y \in E : y = \text{weak} - \lim_{j} T^j x \text{ for some } n_j \to \infty \}.
\]

Then \( T \) has a fixed point in \( K \).

**Lemma 1.3.** ([7]) Let \( E \) be a Banach space with a uniformly Gâteaux differentiable norm, \( C \) be a nonempty closed convex subset of \( E \) and \( \{x_n\} \) be a bounded sequence of \( E \). Let \( \mu \) be the Banach limit and \( z \in C \). Then

\[
\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2
\]

if and only if

\[
\mu_n \langle y - z, J(x_n - z) \rangle \leq 0, \quad \forall y \in C.
\]

where \( J : E \to 2^{E^*} \) is the normalized duality mapping.

**Lemma 1.4.** ([13]) Let \( \{a_n\}, \{b_n\}, \{c_n\} \) be three nonnegative real sequences satisfying the following conditions:

\[
a_{n+1} \leq (1 - \lambda_n) a_n + b_n + c_n, \quad \forall \ n \geq n_0,
\]

where \( n_0 \) is a nonnegative integer, \( \{\lambda_n\} \) is a sequence in \([0,1]\) with \( \sum_{n=1}^{\infty} \lambda_n = \infty \), \( b_n = o(\lambda_n) \) and \( \sum_{n=1}^{\infty} c_n < \infty \), then \( \lim_{n \to \infty} a_n = 0 \).

**Lemma 1.5.** Let \( E \) be a real Banach space, \( E^* \) be the dual space of \( E \) and \( J : E \to 2^{E^*} \) be the normalized duality mapping. Then, for any \( x, y \in E \),
\begin{align}
(1) \quad & \|x + y\|^2 \leq \|x\|^2 + 2(y, j(x + y)), \quad \forall j(x + y) \in J(x + y). \\
(2) \quad & \|x + y\|^2 \geq \|x\|^2 + 2(y, j(x)), \quad \forall j(x) \in J(x).
\end{align}

The purpose of this paper is to prove the following main results for non-self asymptotically non-expansive which improve and extend the corresponding results of Reich [4], [5], Shiojio and Takahashi [7], Wittmann [6], Zhang [8], [9] and Zeng [10].

2. The Main Results

Now, we give the main results in this paper.

**Theorem 2.1.** Let $E$ be a real Banach space with a uniformly normal structure and whose norm is uniformly Gâteaux differentiable. Let $C$ be a nonempty bounded closed convex subset of $E$ and $T : C \to E$ be a non-self asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ satisfying the following conditions:

(i) $k_n \to 1, \ 1 \leq \sup_{n \geq 1} k_n < \sqrt{N(E)}$;

(ii) $\sum_{n=0}^{\infty} (e_n - 1) < \infty$, where

\[ e_n = \frac{1}{n + 1} \sum_{j=0}^{n} k_j \geq 1, \quad \forall n \geq 0. \]

Let $P : E \to C$ be the nonexpansive retraction, $\{\alpha_n\}, \ \{\beta_n\}$ be two sequences in $[0, 1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. For any given $u \in C$ and $n \geq 1$ define a contractive mapping $S_n : C \to C$ by

\[ S_n(z) = (1 - d_n)u + d_n(PT)^nz, \quad n \geq 1, \]

where $d_n = \frac{1}{k_n}, \ \forall n \geq 1, \ t_n \in (0, 1), \ t_n \to 1 \ (n \to \infty)$ and $k_n^2 - 1 \leq (1 - d_n)^2, \ \forall n \geq n_0$, for some nonnegative integer $n_0$. If $z_n \in C$ is the unique fixed point of $S_n$, i.e.,

\[ z_n = S_nz_n = (1 - d_n)u + d_n(PT)^nz_n, \]

then the sequence $\{z_n\}$ defined by (1.7) and the sequence $\{x_n\}$ defined by (1.2) converge strongly to the same fixed point $z \in F(PT)$ if and only if

\[ ||x_n - PT(x_n)|| \to 0, \quad ||z_n - PT(z_n)|| \to 0 \ (n \to \infty). \]
Proof.

Necessity.

If the sequences \( \{x_n\} \) and \( \{z_n\} \) defined by (1.2) and (1.7), respectively, converge strongly to the same point \( z \in F(PT) \), then, by Proposition 1.1, we have
\[
||x_n - PTx_n|| \leq ||x_n - z|| + ||PTz - PTx_n|| \\
\leq (1 + k_1)||x_n - z|| \to 0 \quad (n \to \infty).
\]

Similarly, we can prove that \( ||z_n - PTz_n|| \to 0 \) (as \( n \to \infty \)).

Sufficiency.

For any \( v \in F(PT) \), from (1.7), we have
\[
||z_n - [(1 - d_n)u + d_nv]|| = ||(1 - d_n)u + d_n(PT)^n z_n - [(1 - d_n)u + d_nv]|| \\
= d_n||(PT)^n z_n - v|| \leq d_n k_n ||z_n - v||.
\]

Again, from Lemma 1.5, we have
\[
||z_n - [(1 - d_n)u + d_nv]||^2 = ||d_n(z_n - v) + (1 - d_n)(z_n - u)||^2 \\
\geq d_n^2 ||z_n - v||^2 + 2d_n(1 - d_n)\langle z_n - u, J(z_n - v) \rangle.
\]

From (2.1), we have
\[
2d_n(1 - d_n)\langle z_n - u, J(z_n - v) \rangle \\
\leq ||z_n - [(1 - d_n)u + d_nv]||^2 - d_n^2 ||z_n - v||^2 \\
\leq d_n^2 (k_n^2 - 1) ||z_n - v||^2.
\]

By the assumption that \( k_n^2 - 1 \leq (1 - d_n)^2 \), we have
\[
\langle z_n - u, J(z_n - v) \rangle \leq s_n D^2,
\]
where \( D = diamC \) (the diameter of \( C \)), \( s_n := \frac{d_n(1-d_n)}{2} \to 0 \) as \( n \to \infty \). Therefore, we have
\[
\lim sup_{n \to \infty} \langle z_n - u, J(z_n - v) \rangle \leq 0.
\]

Let \( \mu \) be the Banach limit. We define a mapping \( \phi : C \to [0, \infty) \) by
\[
\phi(x) := \mu_n ||z_n - x||^2, \quad \forall x \in C.
\]
Since $E$ has uniformly normal structure, it is reflexive. Again, since $\phi(x) \to \infty$ (as $||x|| \to \infty$), $\phi$ is continuous and convex, by Mazur and Schauder’s Theorem [19], there exists $x^* \in K$ such that $\phi(x^*) = \inf_{x \in C} \phi(x)$. This implies that the set

$$K = \{ y \in C : \phi(y) = \inf_{x \in C} \phi(x) \}$$

is a nonempty bounded closed convex subset of $C$.

Now, we prove that

$$\bigcup_{y \in K} \omega_w(y) \subset K,$$

where $\omega_w(y)$ is the weak $\omega$–limit set of mapping $PT$ at $y$.

Indeed, for any $y \in K$ and any $p \in \omega_w(y)$, there exists a subsequence $\{m_j\} \subset \{m\}$ such that $p = \text{weak} - \lim_{j \to \infty} (PT) m_j y$. By the weakly lower semi-continuity of $\phi$, we have

$$\phi(p) \leq \lim \inf_{j \to \infty} \phi((PT)^{m_j} y) \leq \lim \sup_{m \to \infty} \phi((PT)^{m} y)$$

or equivalently

$$\phi(p) \leq \lim \sup_{m \to \infty} \mu_n ||z_n - (PT)^{m} y||^2$$

(2.6)

$$= \lim \sup_{m \to \infty} \mu_n ||z_n - (PT)^{m} z_n + (PT)^{m} z_n - (PT)^{m} y||^2.$$

By the condition (1.8), $||z_n - (PT)^{m} z_n|| \to 0$ as $n \to \infty$, we can prove (by using induction) that

$$||z_n - (PT)^{m} z_n|| \to 0, \text{ (as } n \to \infty), \forall m \geq 1.$$

Therefore, from (2.6) and (2.7), it follows that

$$\phi(p) \leq \lim \sup_{m \to \infty} k^2 \mu_n \mu_n ||z_n - y||^2$$

(2.8)

$$\leq \lim \sup_{m \to \infty} \mu_n ||y - z||^2 = \min_{x \in C} \phi(x),$$

i.e., $p \in K$. The conclusion (2.5) is proved. Therefore, the set $K$ satisfies the condition (A). By Lemma 1.2, the mapping $PT$ has a fixed point $z$ in $K$ and $z$ is also a minimal point of $\phi$ in $C$. By Lemma 1.3, we have

$$\mu_n \langle y - z, J(z_n - z) \rangle \leq 0, \forall y \in C.$$
Especially, taking $y = u$, we have

$$ \mu_n \langle u - z, J(z_n - z) \rangle \leq 0. $$

Combining (2.3) and (2.10), we have

$$ \mu_n \| z_n - z \|^2 = \mu_n \langle z_n - z, J(z_n - z) \rangle 
= \mu_n \langle z_n - u + u - z, J(z_n - z) \rangle 
\leq \mu_n \langle z_n - u, J(z_n - z) \rangle \leq 0. $$

Therefore, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $\{z_{n_i}\}$ converges strongly to $z$.

Next, we prove that every subsequence of $\{z_n\}$ converges strongly to the same $z$. Suppose the contrary, there exists a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $z_{n_j} \to p$. It follows from $\|z_n - PT z_n\| \to 0$ as $n \to \infty$ that $p$ is a fixed point of $PT$. By the assumption that the norm of $E$ is uniformly Gâteaux differentiable, the normalized duality mapping $J$ is uniformly continuous on every bounded subset of $E$ from the strong topology of $E$ to the weak$^*$ topology of $E^*$.

Observe that

$$ |\langle z_{n_i} - u, J(z_{n_i} - p) \rangle - \langle z - u, J(z - p) \rangle| 
= |\langle z_{n_i} - z, J(z_{n_i} - p) \rangle + \langle z - u, J(z_{n_i} - p) - J(z - p) \rangle| 
\leq \|z_{n_i} - z\| \cdot \|z_{n_i} - p\| + \|z - u, J(z_{n_i} - p) - J(z - p)\|. $$

Since $z_{n_i} \to z$ (as $i \to \infty$), we have

$$ \lim_{i \to \infty} \langle z_{n_i} - u, J(z_{n_i} - p) \rangle = \langle z - u, J(z - p) \rangle. $$

Similarly, we can prove that

$$ \lim_{j \to \infty} \langle z_{n_j} - u, J(z_{n_j} - z) \rangle = \langle p - u, J(p - z) \rangle. $$

It follows from (2.3), (2.11) and (2.12) that $\langle z - u, J(z - p) \rangle \leq 0$ and $\langle p - u, J(p - z) \rangle \leq 0$. Adding up these two inequalities, we have

$$ \langle z - p, J(z - p) \rangle = \|z - p\|^2 = 0, $$

i.e., $z = p$. This implies that $z_n \to z \in F(PT)$ as $n \to \infty$.

Next, we prove that the sequence $\{x_n\}$ defined by (1.2) converges strongly to $z \in F(PT)$ too. In fact, it follows from (1.2), Lemma 1.5(i) and Proposition 1.1...
that, for any \( n \geq 1 \),
\[
||x_{n+1} - z||^2 \\
= ||x_{n+1} - [(\alpha_nu + (1 - \alpha_n)z] + \alpha_n(u - z)||^2 \\
\leq ||x_{n+1} - [\alpha_nu + (1 - \alpha_n)z]||^2 + 2\alpha_n(u - z, j(x_{n+1} - z)) \\
= ||\alpha_nu + (1 - \alpha_n)\frac{1}{n+1} \sum_{j=0}^{n} (PT)^jy_n - \alpha_nu - (1 - \alpha_n)z||^2 \\
+ 2\alpha_n\langle u - z, j(x_{n+1} - z) \rangle
\]
(2.13)
\[
\leq ||(1 - \alpha_n)\frac{1}{n+1} \sum_{j=0}^{n} (PT)^jy_n - z)||^2 + 2\alpha_n(u - z, j(x_{n+1} - z)) \\
\leq (1 - \alpha_n)^2\frac{1}{n+1} \sum_{j=1}^{n+1} k_j^2||y_n - z||^2 + 2\alpha_n(u - z, j(x_{n+1} - z)) \\
= (1 - \alpha_n)^2e_n^2||y_n - z||^2 + 2\alpha_n(u - z, j(x_{n+1} - z))
\]

Now, we consider the first term on the right side of (2.13). From (1.2), we have
\[
(1 - \alpha_n)^2e_n^2||y_n - z||^2 \\
= (1 - \alpha_n)^2e_n^2||\beta_nx_n + (1 - \beta_n)\frac{1}{n+1} \sum_{j=0}^{n} (PT)^jx_n - z||^2 \\
\leq (1 - \alpha_n)^2e_n^2||\beta_n(x_n - z) + (1 - \beta_n)\frac{1}{n+1} \sum_{j=0}^{n} (PT)^jx_n - z)||^2 \\
\leq (1 - \alpha_n)^2e_n^2\{\beta_n||x_n - x^*|| + (1 - \beta_n)\frac{1}{n+1} \sum_{j=0}^{n} k_j||x_n - z||\}^2 \\
\leq (1 - \alpha_n)e_n^2\{\beta_n||x_n - z|| + (1 - \beta_n)e_n||x_n - z||\}^2 \\
\leq (1 - \alpha_n)e_n^4||x_n - z||^2 \\
\leq (1 - \alpha_n)||x_n - z||^2 + (1 - \alpha_n)(e_n^4 - 1)||x_n - z||^2 \\
\leq (1 - \alpha_n)||x_n - z||^2 + (e_n - 1)M
\]
where \( M = D^2 \cdot \sup_{n \geq 0}(e_n^3 + e_n^2 + e_n + 1) < \infty \) and \( D = \text{diam}(C) \). Substituting (2.14) into (2.13), we have
\[
||x_{n+1} - z||^2 \leq (1 - \alpha_n)||x_n - z||^2 + (e_n - 1)M + 2\alpha_n\langle u - z, j(x_{n+1} - z) \rangle
\]
(2.15)

Since
\[
z_m - ((1 - d_m)u + d_m x_n) = (z_m - x_n) - (1 - d_m)(u - x_n), \quad \forall n \geq 0, \ m \geq 1,
\]
it follows from Lemma 1.4 and (1.7) that
\[ \|z_m - x_n\|^2 = \|z_m - ((1 - d_m)u + d_m x_n) + (1 - d_m)(u - x_n)\|^2 \]
\[ \leq \|z_m - ((1 - d_m)u + d_m x_n)\|^2 + 2(1 - d_m)\langle u - x_n, j(z_m - x_n)\rangle \]
\[ = \|(1 - d_m)u + d_m(PT)^m z_m - ((1 - d_m)u + d_m x_n)\|^2 + 2(1 - d_m)\langle u - z_m + z_m - x_n, j(z_m - x_n)\rangle \]
\[ \leq d_n^2\|(PT)^m z_m - x_n\|^2 + 2(1 - d_m)\|z_m - x_n\|^2 + 2(1 - d_m)\langle u - z_m, j(z_m - x_n)\rangle. \]

Since the normalized duality mapping $J$ is odd, i.e., $J(-x) = -J(x), x \in E$, we have
\[ \langle u - z_n, j(x_n - z_m)\rangle \]
\[ \leq \frac{1 - 2d_m}{2(1 - d_m)}\|z_m - x_n\|^2 + \frac{d_n^2}{2(1 - d_m)}\|(PT)^m z_m - x_n\|^2 \]
\[ = \frac{2d_m - 1}{2(1 - d_m)}\{\|(PT)^m z_m - x_n\|^2 - \|z_m - x_n\|^2\} + \frac{1 - d_m}{2}\|z_m - x_n\|^2 \]
\[ \leq \frac{2d_m - 1}{2(1 - d_m)}\{\|(PT)^m z_m - (PT)^m x_n\|^2 + \|(PT)^m x_n - x_n\|^2\} - \|x_n - z_m\|^2 + \frac{1 - d_m}{2}\|(PT)^m z_m - x_n\|^2 \]
\[ \leq \frac{2d_m - 1}{2(1 - d_m)}\{(k_m\|z_m - x_n\| + \|(PT)^m x_n - x_n\|^2 - \|x_n - z_m\|^2\} + \frac{1 - d_m}{2}\|(PT)^m z_m - x_n\|^2 \]
\[ \leq \frac{2d_m - 1}{2(1 - d_m)}\{(k_m^2 - 1)\|z_m - x_n\|^2 + 2k_m\|z_m - x_n\|\cdot\|(PT)^m x_n - x_n\|^2 \}
\[ + \|(PT)^m x_n - x_n\|^2\} + \frac{1 - d_m}{2}\|(PT)^m z_m - x_n\|^2 \].

Since $C$ is a bounded subset of $E$ and
\[ \|(PT)^m z_m - x_n\| \leq \|(PT)^m z_m - z\| + \|x_n - z\| \]
\[ \leq k_m\|z_m - z\| + \|x_n - z\| \]
and hence $\{(PT)^m z_m - x_n\}$ is also bounded. Letting
\[ S = \sup_{m \geq 1, n \geq 0} \{\|(PT)^m z_m - x_n\|, D\} < \infty, \]
we have
\( \langle u - z_m, j(x_n - z_m) \rangle \leq \frac{2d_m - 1}{2(1 - d_m)}((k^2_m - 1)S^2 + 2 k_m S \lVert (PT)^m x_n - x_n \rVert^2 + \frac{1 - d_m}{2} S^2) \)

(2.16)

From the condition (2.8): \( \lVert x_n - PTx_n \rVert \to 0 \ (n \to \infty) \), by induction, we can prove that

\[ \lVert (PT)^m x_n - x_n \rVert \to 0 \ (n \to \infty) \quad \forall m \geq 1. \]

(2.17)

Again, since \( k^2_m - 1 \leq (1 - d_m)^2 \), it follows from (2.16) and (2.17) that

\[
\limsup_{n \to \infty} \langle u - z_m, j(x_n - z_m) \rangle \leq \frac{2d_m - 1}{2(1 - d_m)}((k^2_m - 1)S^2 + \frac{1 - d_m}{2} S^2) \\
\leq (2d_m - 1) \frac{1 - d_m}{2} S^2 + \frac{1 - d_m}{2} S^2 \\
= (1 - d_m)d_m S^2, \quad \forall m \geq 1.
\]

(2.18)

Hence, for any given \( \varepsilon > 0 \), there exists a positive integer \( n_0 \) such that

\[
(2.19) \quad \langle u - z_m, j(x_n - z_m) \rangle \leq (1 - d_m)d_m S^2 + \varepsilon, \quad \forall n \geq n_0, \ m \geq 1.
\]

Since \( d_m \to 1, \ z_m \to z \ (m \to \infty) \) and \( J \) is uniformly continuous on any bounded subset of \( E \) from the norm topology of \( E \) to the weak* topology of \( E^* \) and hence we have

\[
\limsup_{m \to \infty} \langle u - z_m, j(x_n - z_m) \rangle = \langle u - z, j(x_n - z) \rangle \leq \varepsilon, \quad \forall n \geq n_0,
\]

i.e.,

\[
\langle x - z, j(x_n - z) \rangle \leq \varepsilon, \quad \forall n \geq n_0,
\]

and so

\[
\limsup_{n \to \infty} \langle u - z, j(x_n - z) \rangle \leq \varepsilon.
\]

By the arbitrariness of \( \varepsilon > 0 \), we have

\[
(2.20) \quad \limsup_{n \to \infty} \langle u - z, j(x_n - z) \rangle \leq 0.
\]

Letting \( \xi_n = \max \{ \langle u - z, j(x_n - z) \rangle, 0 \} \) for all \( n \geq 0 \), we know that \( \xi_n \geq 0 \). Now, we prove that

\[
(2.21) \quad \lim_{n \to \infty} \xi_n = 0.
\]
In fact, it follows from (2.20) that, for any give \( \varepsilon > 0 \), there exists a positive integer \( n_1 \) such that
\[
\langle u - z, j(x_n - z) \rangle \leq \varepsilon, \quad \forall n \geq n_1,
\]
and so we have \( 0 \leq \xi_n < \varepsilon \) for all \( n \geq n_1 \). By the arbitrariness of \( \varepsilon > 0 \), this implies that
\[
\lim_{n \to \infty} \xi_n = 0.
\]

From (2.15), we have
\[
\|x_{n+1} - z\|^2 \leq (1 - \alpha_n)\|x_n - z\|^2 + (e_n - 1)M + 2\alpha_n \langle u - z, j(x_{n+1} - z) \rangle \\
\leq (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n \xi_{n+1} + (e_n - 1)M.
\]

In Lemma 1.4, take
\[
a_n = \|x_n - z\|^2, \quad \lambda_n = \alpha_n, \quad b_n = 2\alpha_n \xi_{n+1}, \quad c_n = (e_n - 1)M.
\]
Since \( \sum_{n=0}^{\infty} (e_n - 1) < \infty \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \), we have
\[
\sum_{n=0}^{\infty} \lambda_n = \infty, \quad b_n = o(\lambda_n), \quad \sum_{n=0}^{\infty} c_n < \infty.
\]

This shows that all the conditions in Lemma 1.4 are satisfied. Therefore, we have
\[
\lim_{n \to \infty} \|x_n - z\| = 0.
\]
i.e., \( \{x_n\} \) converges strongly to \( z \in F(PT) \). This completes the proof.

Taking \( \beta_n = 1 \) for all \( n \geq 0 \) in Theorem 2.1, we have the following theorem:

**Theorem 2.1.** Let \( \{z_n\} \) and \( \{x_n\} \) be the sequences defined by (1.7) and (1.3), respectively. If the conditions in Theorem 2.1 are satisfied. Then the sequences \( \{z_n\} \) and \( \{x_n\} \) converge strongly to the same fixed point of \( PT \) in \( C \) if and only if the condition (1.8) is satisfied.

**References**


S. S. Chang
Department of Mathematics,
Yibin University,
Yibin, Sichuan 644007,
P. R. China
E-mail: sszhang_1@yahoo.com.cn

Yeol Je Cho
Department of Mathematics Education and the RINS,
College of Education,
Gyeongsang National University,
Chinju 660-701,
Korea
E-mail: yjcho@gsnu.ac.kr

Y. X. Tian
Department of Computers,
Chongqing Post Telecommunications University,
Chongqing 40065,
P. R. China
E-mail: tianyx@cqupt.edu.cn